A Life Lesson from Calculus

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This past fall, while teaching the theory of limits to my calculus class, we got sidetracked down a trail which culminated in a discussion involving the theologian Paul Tillich. This might seem strange since mathematics and theology are often considered to be polar opposites. However, as we will see by taking the hike together, reality need not be as polarized as it sometimes appears. I've tried to make the trail "mathematically accessible," but if you find yourself getting discouraged, join up again where the equations stop.

First, you need a brief introduction to the mathematical concept of limiting value, or more succinctly, *limit*. Suppose that we have a mathematical expression such as $1/n$. As n takes on different values, the expression $1/n$ changes accordingly. For example, when $n = 5$, $1/n = 1/5 = 0.2$ and when $n = 100$, $1/n = 1/100 = 0.01$. Notice what happens as n becomes an increasingly large number – the value of $1/n$ gets smaller and smaller, closer and closer to 0. Indeed, we can get the value of $1/n$ to be as close to 0 as we want by choosing n to be sufficiently large. (When we take n to be ever larger numbers like $100, 1000$, and $1,000,000$ we say that "n is going to infinity" and write "$n \to \infty$.") Since $1/n$ approaches 0 as n goes to infinity we say that 0 is the limiting value or *limit* of $1/n$ as n goes to infinity, and we write $\lim_{n\to\infty} 1/n = 0$.

Now that you're a pro at limits, let's determine two more to which we will refer later. If you have a calculator, keep it available; it will help you to get a better feeling for them. The first one is $\lim_{n\to\infty} 1.001^n$. ($1.001^5$, for example, is the number obtained by multiplying 1.001 by itself 5 times.) The fun aspect of limits is trying to correctly guess their value before experimenting with a calculator. Can you guess and justify an answer? In this case, the answer is infinity. Why? As we continue multiplying by 1.001, the resulting number gets bigger and bigger, and the product will continue to grow without bound. Experiment with your calculator.

One more preliminary example – an easy one. What is $\lim_{n\to\infty} 1^n$? A little thought will convince you that the value of this limit is itself 1. Why? We are just multiplying 1 by itself over and over again, and, of course, the product will never change from being 1. Thus for every value of n (n is said to be the *power* of 1), $1^n = 1$, so the limit itself is 1.

We now turn to the limit that led my class and me down the philosophical path. Consider the expression $(1 + 1/n)^n$ and again let's try to guess the value of $\lim_{n\to\infty}(1 + 1/n)^n$. First, let's think for a moment about what the expression is asking us to do. As n takes on different values such as 1, 2, 3, etc., what values does the expression give us? Taking $n = 3$ as an example, the expression would be $(1 + 1/3)^3 = (3/3 + 1/3)^3 = (4/3)^3$. When n is 10 it is $(1 + 1/10)^{10} = (10/10 + 1/10)^{10} = (11/10)^{10}$, and when $n = 100$ we have $(1 + 1/100)^{100} = (100/100 + 1/100)^{100} = (101/100)^{100}$.

Without using your calculator, let's guess what the limit might be as $n \to \infty$. Notice that as n changes, two things are happening – two numbers are changing at once. The number $1 + 1/n$ is getting closer to one, and simultaneously $1+1/n$ is being raised to an increasingly greater power. In guessing the value of the limit, our minds limit us to think of only one of these changes at a time. We will see that by necessarily focusing on just one of these changes while ignoring the other, we are led to two different "reasonable" guesses for the limit.

First we focus on the change of $1 + 1/n$. As n gets ever larger, we saw that $1/n$ approaches 0, so $1 + 1/n$ will approach 1. That is, $\lim_{n\to\infty}(1 + 1/n) = 1$. We also saw above that $\lim_{n\to\infty} 1^n = 1$. Since the numbers $1 + 1/n$ get ever closer to 1 as n gets increasingly large, and since $\lim_{n\to\infty} 1^n = 1$, it is certainly reasonable that $\lim_{n\to\infty}(1 + 1/n)^n = 1$.

On the other hand, we can also focus on the changing of the power. We saw in our first example that when we took a fixed number bigger than one (e.g., 1.001) to higher and higher powers, the limit was infinity. Since $1 + 1/n$ will always be bigger than 1, it is then reasonable that as n gets larger and larger, $(1 + 1/n)^n$ will also go to infinity. Thus another good guess is that $\lim_{n\to\infty}(1 + 1/n)^n = \infty$.

The tension between these two possible answers is the heart of the matter. By concentrating on the fact that $1 + 1/n$ is getting arbitrarily close to one, we arrive at one reasonable answer for $\lim_{n\to\infty}(1 + 1/n)^n$. If instead, we give attention exclusively to the fact that numbers bigger than 1 are being raised to ever higher powers, we come to a different conclusion. Both were reasonably obtained; why the disagreement? Because in order to obtain them, in trying to get to the truth of the matter, we over simplified the true situation by focusing on just one part of the expression while ignoring the other.

It was at this point in my lecture when I remembered the imagery Paul Tillich used in his book *Dynamics of Faith* in describing Faith as a Centered Act. Faith, he explains, is an act which lies in the tension between polarities. Similarly, it seems to be the case that truth also often lies in tension between two poles. That is, truth, rather than being pulled to the positive or negative pole of a magnet, may be like a steel ball which is delicately held in suspension between them.

This perception of truth leads to two implications. First of all, we often find ourselves listening to two points of view which lead to diametrically opposite conclusions as is often the case in political/social arguments. (Witness the abortion debate in which we are cajoled to focus on life by one side and rights by the other.) If indeed, truth may be held in tension between these poles, we should be cautious about being forced to choose between the options given.

Before discussing the second implication, let's return to our math problem. Two reasonable arguments led to two different conclusions resulting from focusing on different aspects of the problem. Which answer is correct? It turns out that in order to solve this problem another entirely different method has to be used – a method which bears the name of the 17th century French mathematician L'Hospital. Using his formula and general calculus techniques, it can be readily proved that $\lim_{n\to\infty}(1 + 1/n)^n = e$ where e is a never-ending number (like π) which is approximately 2.72. If you put a large number like 1000 into the expression and use your calculator, you should get an answer close to 2.72. Notice that the true answer – in this case – is a compromise between the other two. It is larger than 1 and less than infinity. More importantly, it was achieved by a means which, instead of building on the previous arguments, rejected them both and found the truth through a new thought process which incorporated both points of view simultaneously.

This observation leads to a second implication of our magnetic model for truth. Namely, not only may truth not reside at the polar extremes which are reached via myopically focusing on just one aspect of the situation, but also, the gaining of truth – if it is to be had – may require a wholly different way of looking at the problem.

Three examples from scripture may illustrate the point. Since wisdom is defined as an "understanding of what is true" we begin with individuals who were noted for their wisdom. King Solomon was asked to adjudicate a case involving two prostitutes and one remaining live infant. One can imagine lower judges trying to determine which was the mother by scrutinizing the testimony and seeking witnesses. Solomon found the truth through a totally different approach. Instead of gathering more evidence, he cleverly made use of his knowledge of human behavior by using the love of the mother to settle the matter. Indeed, creative thinking and a deep understanding of human nature often seem to be essential ingredients of wisdom in the search for truth.

We are also told that God gave Daniel wisdom, and like Solomon, Daniel demonstrated it by making resourceful use of human relationships. Early in his career, Daniel felt the tension between two poles when forced to choose between honoring his God or obeying his king. When ordered by the king to eat rich food for the sake of his health, he and his friends gained permission to demonstrate (via a scientifically controlled experiment!) that their simple diet was healthier than the rich diet prescribed by the king. Thus through innovation and understanding of human nature, Daniel determined a morally true course of action which achieved both objectives.

Although Jesus constantly recast questions and dilemmas in new ways, let's turn to the Apostle Paul for the last example. In chapters 3 – 5 of Romans, Paul provides a lawyer's tight argument for salvation coming totally from God's grace apart from works. But this leads Paul (and presumably the reader) to the question: Why not then continue in sin? After all, we are not saved through our good works, and God's grace can be all the better demonstrated if we continue to habitually sin. So here is a classic two sided argument. Do we focus on God's grace – leading to a sinful lifestyle, or do we focus on the demands of the law – leading to legalism? Paul gets at the truth of the matter by rising to a higher vantage. He explains a principle which incorporates both concerns; namely, through God's grace we have died to sin – it is no longer our master.

So whether it be L'Hospital who provided a way of finding an unexpected answer to a limit problem, or the Apostle Paul who rejected two easy conclusions because of his deeper understanding of our spiritual condition, truth is often not found at the extremes where we are so often inclined and pressed to look.

Instead, in a way that adds to the complexity and richness of life, for the wise it may well be found in tension between two poles.

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