**Math Surprises and Challenges**

**Tim Pennings timothy.pennings@gmail.com**

1. The distance around the earth’s equator is 25,000 miles. How much length must be added to a rope lying on the earth's equator in order to raise it 1 foot above the equator at all points?

**Answer:**Let be the radius of the earth in feet. Then the added length of the rope is

ft. About 6 ft of rope!

1. Take a long rectangular strip of paper and tape the two ends together. Notice that the strip has two sides. If ants can’t crawl over the edge, two ants could live on the two surfaces without ever meeting each other. The strip also has two edges which don’t meet. Cut down the middle of the strip and you’ll get two strips that can be pulled apart. Now put a single twist in the strip before taping it. Run your finger along the surface and you will notice that it has ONLY ONE SIDE. Now two ants WILL meet. It also has just one edge. You have made a Mobius Strip – an example of a “non-orientable surface.” What will happen when you cut the Mobius strip down the middle - what do you get? Try again, but this time start your cut just ¼ of the way from one edge.
2. You have a dresser drawer with 100 black socks and 150 brown socks – all mixed up. You go into the dark room and grab socks – not turning on the light. How many socks do you need to take so that you will be assured of having a pair of socks of the same color? (Pigeon Hole Principle)
3. Assuming no leap year, how many people in a room to be certain that two have the same birthday? Answer: 365 in a room could possibly all have different birthdays. But as soon as you add one more person, that person HAS to have the same birthday as someone else in the room. So the answer is 366. OK. Fine. So then, how many people need to be in a room before the probability of two with the same birthday is greater than 1/2? That is, it is more likely than not that there will be two people with the same birthday? Is it 183 people? (That’s a common answer.)

**Answer:** Suppose there are 5 people in the room. Let’s first find the probability that none of them have the same birthday. It’s a simple counting problem. There are 366 choices for each person’s birthday, so the total number of combinations for birthdays is . How many of these would result in DIFFERENT birthdays? The first person has 365 choices, the second person has 364, the third 363, the fourth 362, and the fifth 361.

Thus there are ways for these five people to have different birthdays. So the probability of no repeated birthdays is the ratio: .

Thus the probability of a repeated birthday is . If you use this same approach you will find that when there are 23 people in the room, the probability of a repeated birthday is greater than ½

1. You are on a motorcycle trip and you come to a hill. It is a one mile along the road to the top of the hill, and another mile to get back to the bottom of the hill. If you travel up the hill at 30 mi/hr, how fast have to go down the hill so that average speed is 60 for the entire hill is 60 mi/hr?

**Answer**: Traveling 60 mi/hr is 1 mi/minute. So for the average speed over the two-mile trip to have an average speed of 60 mi/hour, it must take a total of 2 minutes. But notice that the entire 2 minutes was used in going up the mile hill at 30 mi/hr. Thus, the motorcycle would have to go infinity fast down the hill.

1. 1000 lbs of watermelon is 99% water. They dehydrates to 98% water. How heavy are the watermelons now?

**Answer:** Since the 1000 lbs of watermelon is 99% water, that means that 10 pounds (1%) of it is not water. After it dehydrates that 10 pounds is now 2% of the entire weight. That is Thus lbs!

1. There are 10000 people in town. 1% have a deadly but unnoticeable virus. A test for the virus is 99% accurate. That is, if a person has the virus, the test will fail to indicate it 1% of the time, and if the person does not have the virus, the test will falsely indicate that they do have it 1% of the time. What is the probability that a person has the virus given that the test indicates they have it?

**Answer:** By now you are probably getting wary of your guesses – and with good reason. Let’s analyze: What happens when all 10,000 people are tested. 9900 of the people are virus-free, but the test shows that 99 of them (1%), have the virus. 100 people have the virus and the test correctly shows that 99 of them. So the probability of having the virus is just ½. Hope for the best.

1. Why do we have a musical scale with twelve half steps in an octave? The answer is because of a surprising fact about the twelfth root of 2. From one C to the next (an octave) the higher note vibrates exactly twice as fast as the lower: a 2:1 ratio. Since there are twelve notes between them, each of those notes must be the same ratio higher than its neighbor. So that ratio must be the number that when multiplied by itself 12 times gives 2. That number, , is about 1.06. Multiplying this number by itself, you can see that the frequency of the G (fifth of the chord) is almost exactly in a 3:2 ratio with the root (C). The F (the fourth) is almost exactly in a 4:3 ratio with the root, and the E (third of the chord) is almost exactly in a 5:4 ratio with the root. These nice ratios mean that the wave vibrations match up often. Just as, if one visitor comes every 4 days and another comes every 3 days, they will meet every 12 days. The C-E-G line up every 60 vibrations. That is why C-E, C-F, C-G and C-E-G sound so natural together. It is entirely serendipitous that twelve steps from C to C give such nice results, and is undoubtedly why we use the twelve-tone system. How cool is that.
2. Take a snapshot of the equator of the earth at some point in time. No matter what the temperature around the equator (maybe it passes through volcanoes and refrigerators) it turns out that there will always be (at least) two points on the equator of earth - diametrically across from each other - which have the same temperature.

**Proof**: As you walk around the equator, the temperature may increase or decrease very sharply at certain points, but it will always be continuous. That is, it will never instantly jump from one value to another without passing through the in-between values. Agree?

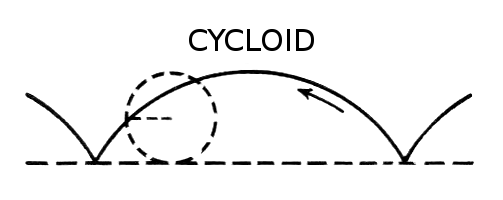
OK. Put your two index fingers ( and ) on the equator diametrically opposite each other. Let’s consider the difference in temp: . If we are done – the temps are the same. So suppose Now slowly move your fingers around the equator until the left index finger is where the right one started, and the right one is where the left one started. Do you agree that now . Since there were no

jumps in between, at some point along the way, which means that the temps are the same.

1. What does the Greek letter stand for in mathematics? Not its value - - but its meaning? Surprising how few students know. (That’s yet another mathematical surprise.) of course is the ratio of the length of the circumference of a circle to the length of the diameter of the circle. No matter the size of the circle, the ratio stays the same. Given that is defined via a circle, it is not surprising that it shows up lots when dealing with circles: the area of a circle, the area and volume of a sphere all involve . What’s surprising is that it also shows up elsewhere in mathematics. For example, take the sum 1 – 1/3 + 1/5 – 1/7 + . . . and multiply it by 4 and you will keep getting closer and closer to . (The proof of that requires calculus - - something to look forward to.)

But there’s even a neater example of where shows up. It’s called the Buffon Needle problem. Randomly toss a one-inch long pin on a piece of paper with horizontal lines spaced one inch apart. Sometimes the needle will cross one of the lines, sometimes not. If it the total number of tosses, and is the number of times the pin crosses a line, then, for a large number of tosses, . Why is that? When a pin is tossed, let be the distance from the lower end of the pin to the line above it, and let be the angle between and radians that the pin makes with a horizontal line. Notice that the total number of ways the pin can land then is represented by a rectangle which has area of . Of this total number of possibilities, the pin will cross the line above exactly when . This corresponds to the area under the graph of on the interval from to In Calculus I class, this area is shown to be . Thus the ratio . Solving for we get the result.

Area is

The final surprise involving concerns where it *doesn’t’* show up. Suppose you have a bicycle wheel with radius of that runs over a firefly at night. Draw the path that the (lit up) firefly make as the wheel makes one complete revolution. That path is special enough that it has its own name - - it is called a *cycloid.* What is the length, of that cycloid?

**Answer:** Let’s get a lower bound and an upper bound for the cycloid. First note that has to be greater than the length of the base which is the length of the circumference of the wheel: . But we can do better than that. The cycloid is also

longer than the two sides of the isosceles triangle with base of and altitude of .

Show that that length is . For an upper bound, note that is less than the length of the three sides of the by rectangle that contains it. The sum of the lengths of those three sides is . So That is, . It turns out – using calculus – that the length of is exactly

Why is this surprising? Can you think of any other measurement involving a circle that *does not* include ?

The cyloid has other surprises. Suppose you wanted to build a slide which got you from point to point as quickly as possible (assume no friction and constant gravity). Draw your best guess. It turns out that even though a straight line from point to point is the shortest *distance,* the path from a cycloid will give you the shortest time. This is called the brachistochrone problem. (brachi means “short” and chrone means “time”.) Equally interesting, if you start anywhere along the cycloid, it will always take exactly the same amount of time to reach the bottom. Christian Huygens used this fact to build a pendulum clock with a pendulum (on an adjustable arm) that ran along a cycloid. Then no matter how high or low the pendulum swung, it’s period (time for an complete back-and-forth) always stayed the same.

The cycloid even has a surprising history. Have you heard of the Bernoulli brothers – Jacob and Johann? They lived in the 17th century and were both great mathematicians and physicists. And like all brothers, they were very competitive. It turned out Johann solved the brachistochrone problem and when he did, he wanted the whole scientific world to know that he was smarter than his brother. So he posed the problem as a challenge to the mathematical community – hoping that his brother wouldn’t be able to solve it. Among the replies, Jacob got one anonymous solution which was by far the most elegant of all of them. The mathematical technique used was clever and quick and elegant. When Johann saw the solution, he immediately announced, “This solution is from Sir Isaac!” (meaning Isaac Newton). When asked how he could be so sure, he replied, “You can tell the lion from the size of the paw.” In other words, Johann know that only Isaac Newton had the genius to give such a beautiful solution. He was right.

1. Suppose you had two great-great-. . . great grandfathers who 200 years ago each invested $10 in the bank for you at 6% interest compounded annually. The first, Abe, kept the principle in the bank, but removed the interest (60 cents) each year and stored it in a piggy bank at home. The second, a bit smarter (or lazier) left the interest in the bank account as well. How much is waiting for you in each account today?

**Answer:**  Since the first account gets $0.60 each year, the total money for you will increase annually: $10, $10.60, $11.20, $11.80, $12.40. $13.00, . . . After 200 years the total will be $10 + (200)($0.60)=$130.

To calculate the bank account for the second ancestor, we proceed as follows:

Year 1:

Year 2:

Year 3:

Year 4:

Year 5:

That’s a full 38 cents more after 5 years by leaving the interest in the bank. What will be the total after 200 years: Year 200:

1. Suppose I have a big hat and an infinite string of billiard balls, each marked with consecutive numbers, 1, 2, 3, . . . I put balls numbered 1 through 10 in the hat and then take out #1. Then put in #11 through #20 and take out #11. Then put in #21 through #30 and take out #21. Get the pattern? After doing this an infinite number of times, do you agree that I have an infinite number of balls in the hat, and an infinite number outside? E.g., #82, #347, and #2048 are in the hat, and #41 and #6591 are outside. No problem. Now suppose that you have an identical hat and set of balls. You do exactly the same in sync with me, except that you draw out different numbered balls. After putting in #1 through #10, you take out #1. After putting in #11 through #20, you take out #2. After putting in #21 through #30, you take out #3, etc. Got it? The only difference between us is the marking on the removed ball. Now the kicker: After an infinite number of times, how many balls are in your hat? Infinite? Nope. None. Don’t believe me? Each ball has a number - name one that is in your hat.

This example shows that things may change when jumping from the world of the finite to the world of the infinite. As long as the number of ball transactions is finite – even a super large finite number, the number of balls in the hats is equal – different numbers on the balls, but equal number of balls. But as soon as we pass from the finite to the infinite (just like the Millennium Falcon going into Hyperspace), things change. Now, suddenly, the second hat has no balls at all. The infinite is mysterious.

1. Notice that this many things can be put in evenly sized groups, whereas this many things cannot. That is why we call “7” a prime number. Prime numbers are the Natural numbers larger than 1 which can only be divided by themselves. Let’s list out the primes less than 100:

Notice that they become more spread out as the numbers grow. That’s to be expected since there are more possible ways to divide big numbers. So it’s reasonable to wonder whether at some point they stop. Is there a largest prime number. After that number, all others can be divided evenly into smaller numbers? 2500 years ago Euclid answered that question: The answer is: There are an infinite number number of prime numbers – there is no largest prime. What is surprising is how beautiful and elegant the proof is.

Maybe the most beautiful proof in all of mathematics:

Background: The most basic thing in all of mathematics are the integers: The mathematician Kronecker famously said of mathematics, “God made the integers, all the rest is the work of man.” And the building blocks of the integers are the prime numbers. In fact, the Fundamental Theorem of Arithmetic says that: Every integer (greater than 1) that is not a prime number can be written uniquely as the product of prime numbers. For example, 30=2x3x5 and 28=2x2x7. Except for changing the order, there is only one way to write an integer as the product of prime numbers.

The Fundamental Theorem of Arithmetic is an example of the most important theorems in mathematics: theorems that guarantee that there is a solution - and what is more - there is only one solution. We want that sort of assurance in many things in life. A person who buys a famous painting wants to know that is really IS authentic, and that it is the only one. A child wants to know that there is one and only one Santa Claus. Many religions claim that the Supreme Being asserted by that religion is the one and only one God. Muslins quote, “There is only one God, and Mohammed is his prophet.” Jews and Christians make similar claims of Jehovah.

So, now here is how Euclid’s proof goes: Suppose that there are only a finite number of primes. In that case, there is a largest prime number – let’s call it N. We now form a new number, M, by multiplying all the primes together and then adding 1: That is, M=2 x 3 x 5 x 7 x 11 x 13 x 17 x 19 x 23 x 29 x 31 x . . . x N +1. Now this new number, M, either is a prime number or it is not a prime number. Agree?

If it IS a prime number, then since it is larger then N, N is NOT the largest prime number afterall. If M is NOT a prime number, then according to the Fundamental Theorem of Arithmetic, M can be written as the product of prime numbers. That’s all well and good. But, NONE of those prime numbers are less than or equal to N. Why?

Notice that 2 x 3 x 5 + 1 = 31 and because of the added 1, 31 can’t be divided evenly by 2 or 3 or 5. There will always be a remainder of 1. Similarly, M can’t be divided evenly by any of the primes up to and including N. Thus, if M DOES have prime factors, those factors are greater than N. Thus we have shown that whether or not M is a prime number, EITHER WAY there must be a prime number greater than N.

There are lots and lots of neat and surprising facts and unanswered questions about prime numbers. For example, notice that many of the prime numbers in our string above come in pairs – just two apart. These are called “twin primes.” Are there an infinite number of pairs of twin primes? No one knows.

Also notice that even numbers are the sum of two prime numbers: Is this true for *ALL* even numbers? No one knows. Some surprises await!

1. Some surprises are deadly. The Pythagoreans were perhaps the original Greek fraternity. It’s no exaggeration to say that they LOVED numbers. They were infatuated by them. The fact that the four rows of bowling pins - with 1, 2, 3, and 4 pins respectively – added up to 10 was significant to them. The harmonics of a harp – with strings of various lengths got them excited. When the ratios of the lengths were 2:1, 3:2, 4:3, and 5:4, for example, then the resulting chords had an especially pleasant sound. (see #15 below). These ratios led to the idea of the rational numbers – numbers created by taking the ratios of integers. They were all beautiful – and the Pythagoreans believed that every possible number represented along a number line was a rational number.

Of course the Pythagorean Theorem was a highlight of their work and discoveries. Yet, that theorem led to trouble. According to the theorem, if you have a right triangle with sides of length 1, then the length of the hypotenuse is . So if you take that distance and mark it off on a number line, you can see just exactly where lay on the number line.

What’s the problem with that, you ask? The problem is, they could not determine the exact value of Apparently, it was close to the rational number 141/100 = 1.41 because . 1414213562/1000000000 is even closer to it, since multiplied by itself gives 1.999999999. Almost there, but not quite. They assumed that the exact value just needed a bit more work to find – maybe a lot more work – but that, whether found or not, for some integers and

Then, according to legend, one of the Pythagoreans made a fateful discovery. He (they were all men) discovered that it was IMPOSSIBLE that could be written as the ratio of integers. How does one discover that something is impossible? We don’t know what

proof he found, but here is clever one: Suppose that it IS possible to write for some integers and Let’s also safely assume that and are the smallest such numbers. (For example, instead of 21/14, we would write 3/2.) Thus and are the smallest positive integers for which Now, make a square with sides of length , and inside the upper left and lower right corners of that square, make two squares with sides of length . (Notice that these two inside squares must overlap. Why?) So, the area in the big square is equal to the sum of the areas of the two inner squares.

Because of that, the area of overlap of the two inner squares must equal the area not included in either inner square. What are those areas? The not-included area has side length of , so the sum of those two square is The length of the side of the overlap square is , so its area is . Since the area of overlap has to equal the area of the excluded region, we then have: But notice what we have just done! We have found TWO SMALLER INTEGERS (name and for which the square of the larger is exactly two times the square of the smaller. That contradicts our legitimate assumption that that and are the smallest such numbers.

The gruesome end of the legend is that the Pythagoreans were so distraught at this discovery, and so wanted to keep this calamitous discovery secret, that they took the discoverer out into a deep lake, tied a couple heavy rocks to his ankles, and tossed him overboard. Pretty severe Greek fraternity hazing!

1. Probably no mathematician in history discovered as great of mathematical surprises as George Cantor. He even surprised himself – with one discovery saying, “I see it but I don’t believe it!” Cantor was interested in infinity. Over the centuries mathematicians had been baffled by the infinite (e.g., #5 above). Does the set of Natural Numbers have more or the same number of elements as the set – or did the question even make sense? Cantor solved the problem by saying that two sets have the same number of elements (the same “cardinality”) if you could find a one-to-one correspondence between the elements of the set. In the example above, each natural number is matched with its square, so they have the same number of elements.

Once he had that down, he defined an infinite set as one which could be put in a one-to-one correspondence with one of its proper subsets (that is, some elements are missing).

Since the Natural Numbers were put into a 1-1 correspondence with a proper subset, it is an infinite set. Since can’t be put into a 1-1 correspondence with any of its proper subsets, it is a finite set.

Then Cantor wondered, “Are all infinite sets the same size? That is, given any two infinite sets, can one always find a one-to-one correspondence between them? Since there are an infinite number of rational numbers between each natural number ( between 0 and 1), Cantor thought perhaps there were more rational numbers than counting numbers. But then he had an inspiration. He arranged the rational numbers in an infinite array – with all the elements in a row having the same denominator, but omitting any numbers that are in an upper row:

1/1 2/1 3/1 4/1 5/1 6/1 . . . .

1/2 3/2 5/2 7/2 9/2 11/2 . . . .

1/3 2/3 4/3 5/3 7/3 8/3 . . . . .

1/4 3/4 5/4 7/4 9/4 11/4 . . . .

1/5 2/5 3/5 4/5 6/5 7/5 . . . .

.

.

Notice we can’t pair these with the natural numbers by matching because then no natural number would get matched to ½. BUT, we CAN pair them all if we go down the diagonals:

By alternating positive and negative, we have just shown that all of the rationals can be counted. This was just the start for Cantor. He showed that in fact there ARE more real numbers than natural numbers. He also showed that for any infinite set, the collection of all of its subsets is a larger size of infinity. Since that set in turn is not large as the collection of its subsets, Cantor proved that there were an infinite number of sizes of infinity.

But finally even Cantor got stuck. He wondered if the infinite numbers were discrete (like the natural numbers), or if they were continuous like the real numbers. That is, given two infinite numbers, is it possible to find another infinite number between them?.

Cantor died in an insane asylum – partially as a result of the inability to solve this problem and validate his ideas. But in 1900 the greatest mathematician of his day, David Hilbert, in the key-note address at the International Congress of Mathematicians, listed 23 problems for mathematicians to solve in the next century. Cantor’s problem was number one. The answer, in 1963, shocked everyone. It was totally unexpected.

1. Yet another counterintuitive result follows from the fact that we can count the rational numbers. First choose a really small number positive number – mathematicians often use the Greek letter epsilon, to represent such a number. Now form an infinite number of intervals of decreasing size, and put a rational number in each one. Notice that adds up to 1 (think of slicing a pie), so the intervals above add up to . That means that you can take ALL the rational numbers of the real line, put each in its own interval, and squeeze them together (without overlapping) into as small of a space as you want. That means that the rest of the real line – the infinite length of it – is made up of irrationals. Another way of saying it is that if you were to throw a dart randomly at the real line, the probability that it would land on a rational number is 0; the probability that it would land on an irrational number is 1. Surprising, eh?
2. Here’s a silly question: What happens when you take look at the long term behavior of the sequence ? Pretty boring – terms just stay 1. Now, what about the sequence ? Well, since 1.001 is larger than 1, as it is multiplied by itself repeatedly, the terms will keep growing larger and larger growing arbitrarily large. So we say that the terms go to infinity. So what then happens to this sequence: . Since the inside of the parenthesis is getting closer to 1, maybe, like the first sequence the terms get closer and closer to 1. But, on the other hand, each of the quantities in the parentheses is greater than 1. So, maybe like the second sequence, the terms go off to infinity. Each has a reasonable argument for it. Which is right? Check it out for yourself. Put in a large number for in . What do you get? You will notice that surprisingly neither 1 nor infinity is the answer. Instead, the terms get closer and closer to the interesting irrational number that we call .
3. What are the five most important numbers in all of mathematics? Most would pick 0 (the additive identity) and 1 (the multiplicative identity). Then of course we need and almost as ubiquitous in mathematics is the other irrational number, . Finally, if we want to go beyond just the real numbers, we need to include the imaginary number . Now, what are all of the different mathematical operations? Well, there is addition (and its counterpart, subtraction). Then there is multiplication (and its counterpart division). And there is exponentiation. How surprising that there is a single short equation that includes *all* of these constants and operations. It is called Euler’s Equation. (You can really impress mathematicians by correctly pronouncing “Euler”: Oiler.) Here it is: This equation can be easily proved using stuff from Calculus II courses. Here’s the essential information.

It turns out that where . Try it on your calculator for some value of x and see that it works. The farther you go, the closer the sum will be to the actual value. Even better, use your graphing calculator or DESMOS (DESMOS is a great friend available on the web) to compare the graphs.

That’s not all. In the same way, and Check these graphs as well. The rest is straightforward:

1. Fibonacci sequence and Golden Rectangle: One way that surprises occur in mathematics is in connections between things that have no nothing apparently in common. Take, for example, the Fibonacci Sequence and the Golden Rectangle. The Fibonacci sequence is the infinite string of numbers: in which the next number is the sum of the previous two numbers. That is, if the sequence is written: , then The Golden Rectangle is a rectangle with the property that if you slice a square off of the rectangle, the remaining rectangle is the same shape (only smaller) as the original. How are these two things related? It turns out that the ratio of consecutive Fibonacci numbers, , as gets large is exactly the ratio of the lengths of the sides of a Golden Rectangle. How curious? Why is that?

The solution only requires simple algebra. Beginning with

divide everything by . Then we get, . Now, as gets large, both of

these ratios approach the same number, let’s call it So . Multiplying both sides by we get and using the quadratic equation, we get .

Notice that you can start with any two numbers you want. So try this as a trick. Ask the first two students in a class to choose any integer from 1 to 100. Next person adds them. Next person adds the last two. Next person adds the next two. Continue. The last person divides the last number by the penultimate number. You can guess the result: 1.62

Now let’s analyze the Golden Rectangle. Letting the short side be length and the longer side , the ratio of the sides is therefore Now, after we cut a square from the rectangle, the remaining rectangle has dimensions so the ratio of the sides is . Since we want these two ratios to be equal, . Multiplying and subtracting, we again get with .

1. Some mathematical surprises are unpleasant. Let’s start this story with a map of the United States. Suppose you want to color the map so that any two states that share a border more than a point (such as Four Corners out west). What is the least number of colors will you need to color the map? Now make up your own fictitious country of (any number of) states. Make up a country that requires just two colors. Just as easy to make up a country that requires three colors. A little more thought is needed to make a simple country that requires four colors. Now put a bunch more thought into it to find a country that requires five colors.

Still thinking? You’ll be thinking for quite a while. Mathematicians have been stumped on that one for over a hundred years. On the one hand, no one could draw a map that required five colors, but no could prove that such a map doesn’t exist either. Then in 1979 a couple mathematicians proved that four colors is all that is ever needed. Upon hearing them give their historic talk, one would expect that the audience would break out in applause. However, their proof consisted of using a computer (with 1000 hours of computing time) to check all the possible combinations. Nothing elegant or clever about it. Just a bunch of tedium. That’s why the room sat quiet. Quiet and strangely disappointed. Mathematics is more than finding the correct answer. It’s about finding beauty. Beauty such as Euclid’s proof of an infinite number of primes, or Euler’s Equation.

1. Since we are talking about beauty, one aspect of beauty is symmetry. Consider the

Taj Mahal or a snowflake for example. Physicists find beauty and symmetry in the world and in the mathematics used to describe it. The physicist James Maxwell was able to “guess” at one of the four equations relating electricity and magnetism just by symmetry with the other three known equations. One surprising place where a beautiful symmetry shows up in mathematics is in the powers of numbers. The first time I saw it, I was amazed and had to prove it for myself (using a method called “proof by induction”):

1. Another famous mathematical object loaded with beauty and symmetry is Pascal’s Triangle. It is formed by starting with a 1 at the top vertex. Then the next row (underneath and symmetric about the vertex is the row “ 1 1 “ The next row is

“1 2 1” The next row is “1 3 3 1” Can you guess the next row? “1 4 6 4 1”

The numbers of each succeeding row are gained simply by adding the two numbers directly above it, so the next row is: “1 5 10 10 5 1”

This simple triangle of numbers has so many interesting properties that there is a periodical journal dedicated to new discoveries. Here are a few to start you out:

1. The first row is , the next row is , the next is , and so on. Figure out how the pattern continues when the row includes double-digit numbers.
2. Add up the numbers in each row – notice that they add to powers of 2.
3. Expand and write out the terms in order of decreasing powers of a. Notice that the numbers in front of the terms are the numbers along the rows of Pascal’s Triangle. This is the most useful property of the triangle.
4. Add up the numbers along diagonals, and you will get Fibonacci’s Sequence

Lots of other hidden treasures as well. Mathematics has been described as the study of patterns - - see what other patterns you can find. Googling it will provide many more.

1. How high can you count on your fingers – with fingers either up or down? Simplest way only gets you to 10. When I was young I discovered a way to go higher. On my right hand, I counted by “1” and when the hand was full (at 5), then I put up one finger of my left hand. That way I was able to count up to 30 using just my fingers. The system I discovered is likely what led to the abacus that is still used in China, Japan, the Middle East, and Russia.

In fact, our 10 fingers is the reason we use 10 symbols {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} in writing the Natural Numbers. THIS was a big surprise to me when I first learned that we didn’t have to use 10. Think about it, “10” means “one group of ten and none left over,” “17” means “one group of ten and seven left over,” “34” means “three groups of ten and four left over.” So what would our number system likely be if we had just three fingers on each hand? Instead of based on ten (called base-ten), it would be based on six (called base-six). The six symbols then would be {0, 1, 2, 3, 4, 5}. So how would we count in base-six? The numbers in increasing order are: 0, 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21, . . . Get the idea?

Now, just for practice, let’s count in base-three: 0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, . . . Does this make sense? Back to base-ten, notice that 100 means a group of “ten-squared” and 1000 means “one group of 10-cubed.” Similarly, in base-three, 212 means “two groups of 3-squared plus 1 group of 3 plus two left over” – which equals twenty three (in base-ten).

None of these other bases have much advantage (or disadvantage) over base-ten. However, base-two DOES have a special importance. That is how computers count. Why? Well, In base-two we would count using only “0”s and “1”s. Here are the numbers: 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, . . . Here, of course, each new place stands for another power of 2, so 1010 stands for 1 group of 2 to the third power plus one group of 2 to the first power which equals ten. Computers must use base-two because electrical switches are either “on” (1) or “off” (0). So ALL the immense calculations on a computer are done using base-two.

Finally, let’s answer our question: How far can you count with your fingers? My own method got to 30 using base-five. If we use base-two, we can count to 1023. Impressive!

1. One of the biggest surprises for me was when my 8th grade teacher told us that . Not just GOT CLOSE TO 1, but actually EQUALLED 1. I didn’t believe it until she showed us this simple and elegant algebra proof:

Let . Then . Subtracting, we get,

1. Sometime surprises come just from playing with numbers. For example, write down any 4 digit number in which all the digits are not the same. Rearrange the digits from largest to smallest and also from smallest to largest. Subtract the smaller from the larger and repeat the process. After at most eight iterations you will arrive at 6174 and then it will stay there forever. (6174 is called a fixed point.) What happens when you try the same thing with three digit numbers? With five digit numbers? Discover your own surprises.
2. Elegant and easy proof of Pythagoreans Theorem
3. If you flip two fair coins and are told that at least one of them is a Head, what is the probability that the other is a Head?

**Answer:** The sample space for flipping the coins is {HH, HT, TH, TT}. Given the info of at least one Head, the sample space is {HH, HT, TH}. So the probability that the other is a Head is 1/3.

1. Why does shift things to the LEFT? Walk a mile in moccasins – diversity training – see things from point of view. What number do you need to put in for so that will see a zero?
2. Another Cultural Diversity question: We saw above that using ten symbols (base 10) to write numbers is not the only way to do it. Likely we use base ten just because we happen to have ten fingers. So even though it is universal and “normal” it is rather arbitrary. However the fact that this many things # # # # # # can be divided into equal sized groups, whereas this many # # # # # # # cannot (except the whole group and singles), is an absolute fact about numbers NOT dependent on habit or culture. How does this relate to our cultural behavior norms. What things do we consider wrong which are absolutely wrong, and what things are just the way we are accustomed to doing it, but other cultures might have different ways?
3. Can you make a square circle? Seems impossible, but not. Let’s think: A circle is defined to be all the points in the plane which are equal distant from a given point. A square, of course, is a four sided figure in the plane with sides of equal length and all intersecting sides perpendicular to each other. The way to do it is to broaden your thinking of distance. When you draw a “regular” circle, the distance that you are using is distance “as the crow flys.” Using this usual distance, the set of all points which are one unit from the origin (using the Pythagorean Theorem) is The set of all points that satisfy that equation is a round circle with a radius of unit length. That is all of those points are exactly one unit from the origin.

However, what if you were driving in a city with equal sized blocks and streets that go north-south and east-west? Then to get from one point to another, you cannot go as the crow flys, but instead must follow the streets. So the distance from one point to another is the total distance east to west added to the total distance north to south. In this case, the set of all points that are unit distance from the origin is given by If you graph all of these points, you will find that they make a square with corners at (1,0), (0, 1), (-1, 0) and (0, -1).

Is this square a “real” circle? Sure is. The only thing that changed was our way to measure distance. These two ways seem very different – why are they both real distances? The answer is that they both satisfy the crucial, defining properties of distance: i) the distance from any point to itself is 0, ii) the distance from any point to another point is a positive number, iii) the distance from A to B is the same as the distance from B to A, and iv) the distance from A to C is less than or equal to the distance from A to B added to the distance from B to C. Does that last one remind you of a property of the sides of a triangle? It is called the triangle inequality.

1. Show that there exist irrational numbers and such that is a rational number.

**Answer:** It can all be shown with . Consider . If this number is rational, we are done. If it is irrational, then which is rational, so we are done.

**Fun and Games:**

1. Can the cop catch the robber on a grid with one beveled corner? Make a grid (any size) with one beveled corner. The cop and robber start one block apart. Taking turns, they each move to an adjacent corner with the robber moving first. The cop catches the robber if s/he lands on the robber’s corner. Can the cop catch the robber?

**Solution:** The only way for the cop to catch the robber is for the cop to first traverse the beveled corner. Then the cop can catch the robber at any of the other three corners.

1. Lay a belt – or a length of string or rope – on a table. Pick up both ends of the belt – one end firmly in each hand, and without letting go of either end, tie a simple knot in the belt.

**Solution:** Tie a knot in your arms (not just crossed, but under and over, then while in that position, pick up the ends of the rope and transfer the knot from your arms to a knot in the belt.

1. Competition between two people. Taking turns, starting with the number 1, each person says the next one, two, or three numbers. The LOSER is the person who says 21.

**Solution:** The first person who lands on 4, 8, 12, 16, or 20 wins, because once on one of those numbers, just play so that you advance to the next multiple of 4.

1. A mathematician is a houseguest of a friend. He asks the friend the ages of his three daughters. The friend replies, “The product of their ages is 36.” The mathematician says,

“That’s not enough information.” So the man says, “The sum of their ages is my house number.” The mathematician checks the house number and says, “Sorry, that’s still not enough information.” So the friend says, “My youngest daughter has blonde hair.” Then the mathematician knows the ages. What are the ages?

**Solution:** First list (in an orderly way) all the possible ages: three numbers whose product is 36: (1, 1, 36), (1, 2, 18), (1, 3, 12), (1, 4, 9), (1, 6, 6), (2, 2, 9), (2, 3, 6), (3, 3, 4)

Now take the sums of each of those respectively: 38, 21, 16, 14, 13, 13, 11, 10. Since the sum (the house number) did not identify the numbers, the sum must have been 13 – which happens twice. Since the youngest has blond hair, there must be only one youngest: that is 1, 6, and 6.

1. Find the next term in sequence: 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, 31131211131221, 13211311123113112211, 11131221133112132113212221,

**Solution:** Just verbalize the string to get the next string writing down each number you say. For example, “312221” would be verbalized, “one three, one one, three twos, one one and so you would write: 13113211. Extra credit: Prove that this sequence will never produce any number higher than 3. Also, add up the numbers in each term: 1, 2, 3, 5, 8, What will the next one be? Notice a pattern? SURPRISE: Not everything is the pattern you are expecting.

1. Classic riddle: A hunter hikes one mile south from his camp. Then hikes a mile east. Then hikes a mile north thereby returning to his camp. What color is the bear? The camp was at the North Pole which is why the hunter was able to return to camp after following those directions. Thus it is a white polar bear. NOW, are there any other points on the globe that also satisfy that riddle: a mile south, then a mile east, then a mile north – hence returning to original spot?

**Solution:** Yes, there are in INFINITE number of such points. Consider the latitude line exactly one mile in circumference close to the South Pole. Now start anywhere exactly one mile north of that latitude line. Then going a mile south, a mile east (which goes exactly once around the circle, ending where you began), and then a mile north gets you back at your original point. Is that it? Nope. Consider the latitude line exactly ½ mile in circumference and do the same, and the latitude line 1/3 of a mile long, etc.

1. Three men go into a hotel and each pays $10 for his room. Then the manager realizes that he has overcharged them, so he gives the bellhop $5 to split between them. The dishonest bellhop pockets $2 and gives them each $1 back. So each man has now paid $9. 9 times 3 is 27, plus the $2 of the bellhop equals $29. Where did the extra dollar go?

**Solution:** The trick of this puzzle is that it perverts your thinking. There is no reason to add 2 to 27 to get 29. That’s meaningless. You can either take 27-2 = 25 to find how much the hotel made, or you can take 25 plus 2 plus 1 plus 1 plus 1 = 30 to get the $30.

1. A fun restaurant puzzle: place three glasses at the vertices of an equilateral triangle so that the distance between any pair is slightly greater than the length of a table knife. Now use three table knives to build a platform on which to rest a fourth glass at the center of the triangle.

**Solution:** Resting one end of each knife on the glass, interweave the three ends of the knives in an under-over weave so that it forms a stable platform for the glass.

1. Suppose an ant on a ceiling of a rectangular room sees a tasty morsel of food on the floor. How can the ant find the shortest path along the ceiling, down a wall, and across the floor to the food?

**Solution:** Make a scaled model of the room out of folded paper, and then unfold the edges allowing the ant to walk a diagonal from point to point.

1. You have twelve numbered billiard balls one of which is of a slightly different weight. (NOTE: You don’t know whether it is heavier or lighter.) Using a balance scale just three times, find the odd ball and determine whether it is lighter or heavier than the others.

**Solution:** This one is complicated. The challenge is to SQUEEZE as much information as possible from each of the weighings. Get started by weighing 1, 2, 3, 4 against 5, 6, 7, 8 and leave 9, 10, 11, 12 off to the side. If they balance, then weigh 9, 10 against 11, 1. That’s the idea. Go from there.

1. Put six pennies in the shape of a cross. Notice that a vertical line passes over four of the pennies, and a horizontal line passes over three pennies. Move one penny so that EACH line passes over four pennies.

**Solution:** Place the bottom penny ON TOP OF the center penny.

1. A knight is in a room with two doors and a guard. On alternate days the guard will always lie, and then always tell the truth. One door leads to freedom and the other to a hungry lion. The knight can only ask one question of the guard to determine the correct door. What is it?

**Solution:** If I were to have asked you yesterday which door leads to freedom, what would you have answered? This question is guaranteed to get one lie. So choose the other door.

1. A boss has three intelligent applicants for a job and wants to find the smartest. So she tells the applicants that she will put a black hat or a white on each of their heads. They can see the other hats, but not their own. Looking around, if they see a white hat, they can raise their right hand. If/when they determine the color of their own hat, they raise their left hand. Then she puts a white hat on each head. They all immediately raise their right hand. Eventually one of them raises the left hand. How did the applicant determine the color of their own hat?

**Solution:** The winner reasoned, “If I had a black hat, then either of the other two (intelligent) applicants – by seeing the hands all raised – would easily be able to deduce that they had a white hat. Since they haven’t made that simple deduction, my hat must be white.

1. Make a three by three grid of nine dots (in the shape of a square). How many straight line segments – connected to each other – are needed to pass through all of the nine dots?

**Solution:** It’s easy to do it in five segments. It can be done in four segments by extending your line outside of the confines of the square grid of dots. Try it. As I understand, this was considered the correct answer for years on IQ tests. Then some child realized that it could be done with only three lines. It’s because a “dot” has some dimension to it (unlike a point). So take a line that passes through the top of the left top dot, the center of the top center dot and the bottom of the top right dot. Keep going as far as needed so that on the return trip the next line goes through the top of the first, the middle of the second, and the bottom of the last dot in the second row. Continue to get the last row of dots connected with a third line.

1. A parking lot has numbers painted in the parking spaces. You drive your car in, start walking away from your car and remember that you need to remember where it is. Looking back towards the front of your car, you see: **16 06 68 88** Car **98**. What number is under your car?

**Solution:**  You are parked in spot number 87. The numbers are written to be viewed for drivers coming into the parking spaces.

1. Here is a “proof” that . Can you find the mistake?

Let

So .

So

So

So

So

So

**Solution:** Since , so in the fourth step, you were dividing by .

1. Write down any positive integer whose digits are not all identical. Rearrange the digits and then subtract the smaller of the two numbers from the larger. Now circle any digit in the resulting number (except 0 which is already a circle). Tell me the other digits (in any order) and I’ll tell you the digit you circled.

**Solution:** A bit of modular arithmetic reveals that when a permutation of one number is subtracted from another, the sum of the digits in the difference is a multiple of nine. So if the player gives you the numbers: 2, 5, 8, 1, 4, 4, then since they add up to 24, the circled number must be 3, since 27 is a multiple of 9.

1. Mind reading: Ask someone to choose a positive integer, multiply it by 9, and then add 4. Add the digits of the resulting number together, and add the digits of the resulting number together - - continue until s/he gets to a one digit number. Then find the corresponding letter from the alphabet (A is 1, B is 2, etc.) Then think of a country that begins with that letter, and finally, think of an animal that begins with the second letter of that country. You then read their mind and name the animal.

**Solution:** If the calculations are done correctly, the resulting number will always be 4. So the letter is D. USUALLY the country is then Denmark (although Dominican Republic, Djibouti are other options). Then the animal picked is Elephant (although eel, emu, eagle . . . are other options.) Not a guarantee, but usually works. If multiple people, don’t choose a “wise-guy.”

1. Cut a hole in a 3 in by 5 in recipe card that you can crawl through.

**Solution:** (Hard to explain without demonstrating.) Fold the card in half and then make cuts (alternating the direction). Then cut along the fold.

1. A fun challenge at a restaurant with a glass of water/pop and a see-through straw: Just using your hand (no suction from mouth), get the liquid to climb up the straw to a level considerably higher than the level of the liquid in the glass.

**Solution:** Raise the straw out of the liquid. Place finger over top of straw. Lower the straw to the bottom of the glass and then raise it just slightly off the bottom. Now, when you remove your finger from the top, the liquid will quickly rush into the straw and the momentum will carry it well above the level of the liquid in the glass. Quickly place your finger back on the top of the straw when the liquid reaches the high point.

Practice a bit, and you’ll do it so fast that others can’t see how you are doing it.

1. Symmetric letters: Find the capital letters in the alphabet which are vertically symmetric (A H I M O T U V W X Y). Then write words and names vertically using just these letters. How long of words can you find? TIMOTHY is a good one. Do the same for horizontally **symmetric letters.**
2. A person’s 12 ½ birthday occurs 183 days after their 12th birthday. Use this same idea to find your birthday, birthday, etc. (Choose an interesting number that the students will soon by approaching.
3. Drawing big circles on board: Standing with your back to the chalkboard, and your unbended arm (with chalk) extended as far down as possible while still on the board, turn your body towards your unbended arm and it will sweep out a nearly perfect circle. A great crowd pleaser!
4. Martian Number game: Place a collection of objects on a waist-high table which everyone has gathered around. Have your partner leave the room while a person quietly provides a number from 0 to 10. Make some insignificant changes to the position of the objects and ask your partner to return. By looking at the table, s/he will be able to identify the number.

**Solution:** After rearranging the objects take a standing position leaning slightly over the table and with your hands inconspicuously placed on the table. Your outstretched fingers (the others casually closed) will reveal the number.

1. Proof by rotation: Here is a fun “proof” involving limits: . Rotating each CCW gives . Subtracting 8 from each gives , and then rotating CW provides
2. Hair Fight. After class fun: Ask a couple volunteers to provide a two inch strand of hair – requiring one thick (wavy/curly) strand and one straight thin (usually blond) strand. Pour a small puddle of water on a flat table and invite the students to watch carefully to see what the hairs do. As all are gathered closely around. WHAM! Slam you hand into the middle of the puddle. Instant laughter.
3. The game 24: Each team finds four numbers from 1-9 from which one can obtain the number 24 through adding, subtracting, multiplying, and dividing those four numbers. They give their numbers to the other team simultaneously. First team to solve, wins. Examples:

1, 4, 7, 7: (solution: (7-4)(7+1)=24); 8, 2, 9, 3: (solution: (8/2)(9-3)

1,3,5,7: (solution: (7+5)(3-1), (5+1)(7-3)); 1,2,3,4: (solution: (1+3)(2+4))

3,5,8,9: (solution: (9(8-5)-3); 4,5,6,7: (solution: ((6-4)(7+5))

For experts: 5, 5, 5, 1: (5(5-1/5))

1. Who has faster reflexes? Invite a fun-loving (tolerant) student to challenge you to a contest of reflexed. Have him sit on the floor with legs extended on either side of a small puddle. You crouch in front of him with a rag. Give him two pens to jab at your hands as you try to wipe up the water. On the count of three, toss the rag away, grab the outstretched legs and pull him through the puddle – wiping up the water.
2. Cut a banana in half without peeling it. In advance use a needle and thread to cut the banana. Pierce the peel and keep the needle close to the inner peel. When it exits, pierce in that hole and continue until you exit the original hole. Pulling the threat taught will slice the inner banana in two pieces with only four small holes in the peel. Do the trick fairly soon before the holes discolor.
3. Fold a 11 in by 4 in piece of paper in half so that it is 5.5 by 4. Draw a simple bird on the top with wings up and beak open. On the sheet underneath, draw a similar bird but with wings down and beak closed. Curl the top piece of paper and then use a pencil to rapidly uncurl it - - you will see the illusion of a flying bird.
4. Cross arms in front of you and clasp hands together – interweaving the fingers. Pull hand under and towards the body and then up to your face. Extend index fingers and place the back of the fingers on either side of your nose. Unclasp and separate hands WITHOUT squeezing your nose.

**Solution:** The ability to separate hand without squeezing your nose depends on how the fingers were interlaced. If it didn’t work, situate the other hand’s fingers on top.

1. Read this sentence aloud, and then have the person count the “f”s in the sentence: FINISHED FILES ARE THE RESULT OF YEARS OF SCIENTIFIC STUDY COMBINED WITH THE EXPERIENCE OF MANY YEARS.

**Solution:** Most often the person will count three “f”s - - missing the “of”s because they are silently counting the “f” sounds and hence miss the “v” sound from “of”.

1. Random Sequences: Leave the room. One person flips a coin 30 times and register the string of H’s and T’s for the heads and tails. Everyone else write down a 30 character random string of H’s and T’s. You return and identify the sequence from the coin-flipper.

**Solution:**  This doesn’t always work – but amazingly often. You simply find the longest string of consecutive Heads or Tails. This works because we don’t have a true sense of randomness. True randomness includes some long strings.

1. Have students determine the value of by using a long length of rope in a parking lot or open field. With one student standing and rotating in the center, the other paces out uniform steps around the perimeter. Then walks across the circle passing through the center. The ratio of number of steps is .
2. Put a large collection (20 or so) of varied objects on a table. Invite students to separate the objects into groups according to some criteria. Others have to guess the criteria used. (Not allowed to put “yellow” in one pile and “paper” into another pile and “large” into another pile. Why: Some object could satisfy all three of those. Instead, if “color” is the criteria, then all are grouped by color. If “material” is criteria, then all are arranged by the material they are made of. This is a very useful mathematical idea: equivalence classes – which are central to abstract algebra and higher mathematics.
3. Since . How can we then find the area of a circle? By cutting the circle into thin pie slices and stacking them into a rectangle by alternating the direction, the rectangle will have a base of length and a height of . So the area is .
4. Quickly add the numbers 1 through 100.

**Solution:** Write Then, underneath, write it in the reverse order: . Adding the two rows together, we get:

1. How is heat transferred across space? Conduction, Convection, and radiation. Nice words, but how can they be illustrated? Try this: Ask the students to transfer a pile of tennis balls in one box to another box across the room. Three ways to do it: i) Have students form a line and pass the balls hand to hand (conduction), ii) Have students carry a handful of the balls from box to box (convection), iii) Have the students toss the balls from one box to the other (radiation).
2. Position, velocity and acceleration: Mark off a point in a long hallway and designate the positive and negative sides of it. Then invite students to walk in such a way so that various combinations of position, velocity, and acceleration are positive and negative.
3. Twenty Question Game: Have students ask you questions that you answer with a “yes” or “no” to determine the identity of a person. Unbeknownst to them, you decide beforehand to answer the questions according to some sequence, such as YNYYNNYYYNYNNYYYNYNY. They determined the person by the questions they were asking. This illustrates how we have come to learn about our universe. The universe we live in is, to some extent, dependent upon the questions that we have asked.
4. You flip two coins and a partner looks at them and tells you that at least one of them is a Head. What is the probability that both are heads?

**Solution:** The sample space consists of HH, HT, TH, and TT. Given that at least one is a head, restricts the sample space to HH, HT and TH. Thus the probability that both are heads is 1/3.

1. Given the ten sequences 0 0 0, 1 1 1, 2 2 2, 3 3 3, . . . 9 9 9, use mathematical symbols (but NOT including any digits or decimal points) to make each of the ten sequences equal 6.

**Solution:**

1. Spinning button: You need a heavy big button (at least ½ in diameter – bigger the better) and a length of thread about three feet long. Put the thread through two holes of the button and tie the ends so that you can suspend the button between the index fingers of your two hands. Twirl the botton a couple dozen times. Then by pulling and relaxing the thread, you can get the button to twirl rapidly alternating directions.
2. Moving Rubberband: Starting with a rubber band on first two fingers (while folded), unfold fingers to instantly make it move over to the other two fingers.

Abstract for teachers:

George Polya said that the essence of mathematics is solving problems. In this highly interactive session, participants see a wide collection of mathematical puzzles and challenges which use elementary mathematics, but give rise to

surprising results. A full list including solutions will be provided. These can be included in lectures or used "before the bell rings" to get students engaged in mathematical fun and games. Although fun and fast, there are deep lessons to be learned about the power and mystery of mathematics and the limitations of our intuition.

Abstract for students:

George Polya said that the essence of mathematics is solving problems. This highly interactive talk engages and surprises with a wide collection of mathematical puzzles and challenges which use elementary mathematics to give counterintuitive and mindboggling results showing the power of mathematics and creative thought.