Infinity and the Absolute : Insights into Our World, Our Faith and Ourselves

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I. Introduction

What has infinity to do with faith? Although the connections are not obvious, they are hinted at both by writers of scripture and mathematics. For example, the writer of Ecclesiastes declares: "He has put eternity into man's mind . . . " Likewise we read in Plato's Republic [VII, 527], "The knowledge at which geometry aims is the knowledge of the eternal." Hermann Weyl agrees, " . . . [Mathematical inquiry] lifts the human mind into closer proximity with the divine than is attainable through any other medium. *Mathematics is the science of the infinite*, its goal the symbolic comprehension of the infinite with human, that is finite, means."[1]

The infinite has intrigued me ever since my ninth grade math teacher proved that $0.999\ldots = 1$.[2] Although my thoughts have matured somewhat since that time, I still find the concept just as perplexing. In fact, the more I read and think about infinity and how it relates to questions of faith, the more I question my ability "to know" because I become increasingly aware of the limitations of my own mind. On the other hand, the very activity of contemplating the infinite and endeavoring to understand its connections to the world outside of me and the faith within me, serves me by clarifying my position in the scheme of things. Thus in grappling with the infinite, the mind is at once humbled by its inability to fully understand, while enriched by the very attempt to understand.

David Hilbert, one of the great mathematicians of the early twentieth century, exclaimed:

> The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite . . .[3]

[1] Phillip J. Davis, Reuben Hersh, *The Mathematical Experience*, Boston: Birkhauser, 1981, p. 7.

[2] The proof is simple. Just let $N = 0.999\ldots$. Then $10N = 9.999\ldots$. Subtracting the first equation from the second leaves $9N = 9$, so $N = 1$.

[3] Eli Maor, *To Infinity and Beyond*, Boston: Birkhauser, 1987, p. 15.

We begin to appreciate the difficulty of clarification already in noting the definitions found in the American Heritage dictionary. The first is: "Having no boundary or limits." However, this is easily seen to be inadequate; an ant crawling on a globe encounters neither boundaries nor limits, yet we surely do not consider the surface of a globe as being infinite in extent. Other definitions include being immeasurably large, having endless duration, and the mathematical definition given by Georg Cantor which we discuss later.

We begin with a brief historical overview of the concept of the infinite, especially concerning its relationship to God and the universe. We then discuss the definition of infinity given by Cantor, and his underlying religious beliefs that caused him to believe in the existence not only of infinite numbers, but also of an infinite universe and an Absolute Infinite which he associated with the mind of God. We end with topics related to the infinite including the way we view our earth, our mortality, the existence of absolute truth, and the infinity of God.

II. Historical Survey

Anaximander (611-547 B.C.), the Greek philosopher and astronomer, introduced infinity (especially its use as a divine modifier) to the Western world. He considered the cause of the universe to be "*to apeiron*," which involved being indeterminate, unbounded, inexhaustible, and everlasting. As R. Rucker explains, "*Apeiron* was a negative, even pejorative word . . . [It] need not only mean infinitely large, but can also mean totally disordered, infinitely complex, subject to no finite determination."[4] Pythagoras, who accepted nothing more complicated than the natural numbers ({1,2,3, . . . }) and discrete atoms, rejected the notion altogether. Likewise Plato, who associated *apeiron* with imperfection, believed the Ultimate Good must be finite and definite.[5]

Although Aristotle believed that the First Unmoved Mover had infinite power, his was also a finite world since, in his words, " . . . being infinite is a privation, not a perfection but the absence of a limit . . . "[6] However Aristotle also recognized *apeiron* in the endless duration of time and the divisibility of space. Aristotle solved the dilemma by claiming that time and space were *potentially infinite* (a finite collection which gets arbitrarily large) as opposed to an *actual infinite* which exists as a finished thing. As such, Aristotle was the first to clearly differentiate between two ways of thinking about infinity. As A.W. Moore explains,

[4] Rudy Rucker, *Infinity and the Mind*, Boston: Bantam, 1982, p. 3.

[5] David L. Balas, O. A. Cist, "A Thomist View on Divine Infinity," In *Infinity*, ed. by Daniel O. Dahlstrom, David T. Ozar, and Leo Sweeny, S.J. Volume VL, Proceedings of the American Catholic Philosophical Association. Washington, DC: The Catholic University of America, 1981, p. 91.

[6] Rudy Rucker, p. 3.

> Two clusters of concepts [of the infinite] dominate, and much of the dialectic in the history of the topic has taken the form of oscillation between them. Within the first cluster we find: boundlessness; endlessness; unlimitedness; immeasurability; eternity; that which is such that, given any determinate part of it, there is always more to come; that which is greater than any assignable quantity. Within the second cluster we find: completeness; wholeness; unity; universality; absoluteness; perfection; self-sufficiency; autonomy. The concepts in the first cluster are more negative and convey a sense of potentiality. . . . The concepts in the second cluster are more positive and convey a sense of actuality.[7]

The last influence Greek philosophy would have occurs over 500 years later. Plotinus (205-270 A.D.), who ushered in Neoplatonism, believed in an underlying existence which is distinct from – yet sustains – the world we experience. This underlying reality was the infinite God.[8] Thus Plotinus broke new ground in adopting the belief that God was infinite (in the second sense described above); hence for the first time associating infinity with perfection.

Whereas the Greek thinkers were reluctant to associate infinity with divine perfection, some of the early Christian writers did so readily. These include Philo, Irenaeus, Clement of Alexandria, and Gregory of Nyssa – who was probably the first Christian thinker to expound the infinity of God.[9] Possibly this difference in thought was tied to their respective concepts of time. The Greeks thought of time as being circular and thus saw themselves as being entrapped by it. Eternity, for them, was qualitatively distinct from time – it was timelessness. In contrast, the primitive Christian church inherited from Judaism the belief that time was like an upward-sloping line – moving towards a consummation. Eternity is an attribute of God, and "our time" is but a portion of eternity – of God's time.[10]

Augustine, who sought to integrate Neoplatonism with Christianity, argued for an infinite God who could think infinite thoughts. In his words,

> Every number is defined by its own unique character, so that no number is equal to any other. They are all unequal to one another and different, and the individual numbers are finite but as a class they are infinite. Does that mean that God does not know all numbers, because of their infinity? No one could be insane enough to say that.
>
> Never let us doubt, then, that every number is known to him 'whose understanding cannot be numbered'. Although the infinite series of numbers cannot be numbered, this infinity of numbers is not outside the comprehension of him 'whose understanding cannot be numbered'. And so, if what is comprehended in knowledge is bounded within the embrace of that knowledge, and thus is finite,

[7]A. W. Moore, *The Infinite*, New York: Routledge, 1991, p. 1-2.
[8]Ibid., p. 45.
[9]David L. Balas, O. Cist, p. 92.
[10]Oscar Cullmann, *Christ and Time*, London: SCM Press LTD, 1952, p. 51-65.

> it must follow that every infinity is, in a way we cannot express, made finite to God, because it cannot be beyond the embrace of his knowledge.[11]

Augustine's influence notwithstanding, the next eight centuries did little to further the concept of an infinite God. This may be because the Biblical record itself, while speaking of God's power (Genesis 17:1; John 1:3), eternity (Genesis 21:33; Deuteronomy 32:40; Job 36:26), omnipresence (Deuteronomy 4:39; Psalms 139:7-12; Jeremiah 23:24), transcendence of location (Job 11:7-8), otherness from creation (Is 46:9), and transcendence from human understanding (Job 36:26; Isaiah 55:8-9; Romans 11:33; Ephesians 3:8), never states explicitly that God is infinite. (However, see Psalms 147:5.)

Until the 13th century, divine infinity was a property not so much of God's essence as of His relationship to creation. Augustine for example uses the word "infinitum" interchangeably with God's eternity, omnipotence, incomprehensibility by human minds, and transcendence of location.[12]

It was Thomas Aquinas who first argued that God's nature was infinite in and of itself and not merely in relation to the created world. Aquinas asserted (roughly) that it is matter which limits a being, and so a perfect being which is subsistent in itself (not depending on matter) was then necessarily infinite. So God's infinite nature is inextricably linked to His perfection. Interestingly, Duns Scotus, who followed closely on the heels of Aquinas, agreed that God was infinite, but his reasoning went in the reverse direction. Aquinas maintained that since God was infinite in entity, it followed that His intellect, power and will were then necessarily infinite. Scotus, on the other hand, argued that the infinity of God's will and intellect implied that the divine entity was infinite as well.[13]

Another intriguing argument for God's infinite nature provided by Aquinas is that our "intellect . . . extends to the infinite in understanding. . . . But this ordination of the intellect would be in vain unless an intelligible existed."[14]

On the other hand, Aquinas believed that the world was finite. We find in the *Book of Wisdom*, "Thou hast ordered all things in measure, and number and weight."[15] So Aquinas concludes,

> . . . every kind of multitude must belong to a species of multitude. Now the species of multitude are to be reckoned by the species of numbers. But no species

[11] Michael Hallett, *Cantorian set theory and the limitation of size*, Oxford: Clarendon, 1984, p. 35-36.

[12] Leo Sweeney, Presidential Address: Surprises in the History of Infinity from Anaximander to George Cantor. In *Infinity*, ed. by Daniel O. Dahlstrom, David T. Ozar, and Leo Sweeny, S.J. Volume VL, Proceedings of the American Catholic Philosophical Association. Washington, DC: The Catholic University of America, 1981, p. 8.

[13] Ibid., p. 11-16.

[14] David L. Balas, O. Cist, p. 94.

[15] Michael Hallett, p. 22.

> of number is infinite, for every number is multitude measured by one. Hence it is impossible that there be an actually infinite multitude, either absolutely or accidentally. Furthermore, multitude in the world is created, and everything created is comprehended under some definite intention of the Creator; for no agent acts aimlessly. Hence everything created must be comprehended under a certain number. Therefore it is impossible for an actually infinite multitude to exist, even accidentally.[16]

Consequently, Aquinas infers that mathematics must concern itself only with the potentially infinite. For God represents the only actual infinity, and God is not an object of mathematical study.

As the scientific revolution transformed the way the world was viewed, the notion of the infinite was changed as well. Descartes reflects the traditional view (where infinity is associated with perfection) when he writes,

> My notion of the infinite is somehow prior to that of the finite. . . . For how would it be possible for me to know that I doubt and that I desire – that is, that I lack something and am not all perfect – if I did not have in myself any idea of a being more perfect than my own. . . .[17]

Yet less than fifty years later, John Locke states,

> Finite and infinite seem to me to be looked upon by the mind as the modes of quantity, and to be attributed primarily in their first designation to those things which have parts and are capable of increase and diminution, by the addition or subtraction of any the least part. . . . It is true that we cannot but be assured, that the great God, of whom and from whom are all things, is incomprehensibly infinite; but yet when we apply to that first and supreme Being our idea of infinite, in our own weak and narrow thoughts, we do it primarily in respect of his duration and ubiquity; and I think, more figuratively, to his power, wisdom and goodness.[18]

D. Schweickart emphasizes,

> The new element, of course, is the primary association of infinity with non-metaphysical, purely mathematical, *quantity*. To be sure, the mathematical aspect of infinity has been recognized at least since Zeno, but by the late seventeenth century it has begun to appear to many that this is the *essential* nature of infinity, that other usages are only metaphorical. This shift in meaning is hardly surprising. As Husserl has emphasized, the decisive conceptual innovation of Galilean science is the quantification of quality.[19]

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[16] Ibid., loc. cit.

[17] David Schweickart, A Marxist Perspective on the Human Person. In *Infinity*, ed. by Daniel O. Dahlstrom, David T. Ozar, and Leo Sweeny, S.J. Volume VL, Proceedings of the American Catholic Philosophical Association. Washington, DC: The Catholic University of America, 1981, p. 100.

[18] Ibid., loc. cit.

[19] Ibid., loc. cit.

Even when the notion of the infinite was narrowed to the merely quantitative by the scientific community, it was still not a precisely defined term. Thus the men who ushered in the scientific revolution had their problems coming to grips with infinity as well. For example, the notion of the "infinitesimal" – a quantity larger than zero, but smaller than any finite number – was used extensively and successfully in the effort to describe mathematically the motion of the planets. Yet no one could explain – or apparently quite understand – what an infinitesimal was. This intellectual "fumbling with the ball" was not lost on Bishop Berkley, who in 1734 entitled his book,

> *The Analyst, Or A Discourse Addressed to an Infidel Mathematician. Wherein It is examined whether the Object, Principles and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith. "First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's Eye."*

("The Infidel" was probably Edmund Halley who helped Newton publish the *Principia* and also is said to have persuaded a friend of Berkeley's of the "inconceivability of the doctrines of Christianity."[20])

The use of infinitesimals was replaced by the limit process which is still used today in the calculus. However the mystery surrounding the infinite – especially concerning the existence and properties of the actual infinite as opposed to Aristotle's potential infinite – was not yet solved. Even Gauss (1777-1855), the prince of mathematics, wrote to a friend,

> As to your proof, I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a figure of speech; an abridged form for the statement that limits exist which certain ratios may approach as closely as we desire, while other magnitudes may be permitted to grow beyond all bounds. . . .
>
> . . . No contradictions will arise as long as Finite Man does not mistake the infinite for something fixed, as long as he is not led by an acquired habit of the mind to regard the infinite as something bounded.[21]

It took the mind of Georg Cantor (1845-1918) to finally unravel the infinite knot. His discoveries were truly revolutionary and initially met with strong resistance from the mathematical community. Particularly strong was the attack from his former teacher, Leopold Kronecker (1823-1891) who did not accept any mathematics which was not directly based on the integers. (Kronecker's most famous quote is: "God made the integers; all the rest is the work of man.") Kronecker may also have been jealous of his former student who was "slaying his ten-thousands." Whatever the case, the opposition led to deep bouts of depression in

[20]Phillip J.Davis, Reuben Hersh, p. 325.

[21]Tobias Dantzig, *Number: The Language of Science*, New York: Macmillan, 1954, p. 211-212.

Cantor's later life, and he eventually died in a mental institution. However, the resistance was not unexpected; even Cantor, upon making one discovery exclaimed, "I see it, but I don't believe it."

However Cantor's corner was by no means empty. Hilbert was an enthusiastic proponent who claimed that Cantor's theory was "the most astonishing product of mathematical thought, one of the most beautiful realizations of human activity in the domain of the purely intelligible."[22] To the critics he responded, "No one shall expel us from the paradise which Cantor has created for us." Bertrand Russell said Cantor's work was "probably the greatest achievement of which our age can boast."[23]

It is to Cantor's discoveries that we now turn.

III. Cantor

Cantor's discoveries concerning the infinite are tied in with his development of set theory. A *set* is just a collection of "things." Cantor described a set as a *Many that allows itself to be thought of as a One*. He maintained that a set[24] – even a set with an infinite number of elements – must be regarded as a totality. In an essay appearing in 1883 Cantor wrote,

> It is traditional to regard the infinite as the indefinitely growing or in the closely related form of a convergent sequence, which it acquired during the seventeenth century. As against this I conceive the infinite in the definite form of something consummated, something capable not only of mathematical formulations, but of definition by number. This conception of the infinite is opposed to traditions which have grown dear to me, and it is much against my own will that I have been forced to accept this view. But many years of scientific speculation and trial point to these conclusions as to a logical necessity, and for this reason I

[22]Morris Kline, *Mathematical Thought from Ancient to Modern Times*, New York: Oxford University Press, 1972, p. 1003.

[23]Eli Maor, p. 63.

[24]A brief explanation of some mathematical terminology may be helpful here. As already mentioned in the text, a *set* is just a collection of things. We often use parenthesis to identify together the members of a set. For example, let S be the set $\{3, 5, 6, 7, 9\}$. 7 is said to be an *element* of S. One set is a *subset* of a second set if every member of the first set is also in the second set. If the first set leaves out some members of the second set, then it is called a *proper subset*. For example, the set T = $\{3, 6, 9\}$ is a proper subset of S.

Sets can include things other than numbers. Consider the set of all the names of days of a week, or the set of all letters in the alphabet, or the set of people over 100 years old. The set of people over two hundred years old has no elements in it. The set with no elements is called the empty set and is denoted $\{\}$.

We call the set of numbers represented by the number line the *real numbers*, denoted by **R**. The interval [a, b] denotes the set of real numbers between (and including) a and b. An important subset of the real numbers is the set of counting numbers or *natural numbers*. This is the set consisting of 1, 2, 3, etc. We let **N** represent this set of numbers, so we can write $\mathbf{N} = \{1, 2, 3, \ldots\}$. The *integers* is the set of all natural numbers together with their negatives and the number zero. The set of all the possible ratios of integers (like 2/3) gives rise to the *rational numbers*. The rational numbers can, in turn, be enlarged to the set of *algebraic numbers* by throwing in numbers like $\sqrt{3}$ and $\sqrt[5]{8}$. Finally, by filling in all the remaining holes in the number line with numbers like $\pi \approx 3.1415926\ldots$, one obtains the real numbers again.

> am confident that no valid objections could be raised which I would not be in position to meet.[25]

Thus Cantor accepted the notion of the *actual infinite* whereas the mathematical community to that point had never been more daring than to accept Aristotle's *potential infinite*. In fact, Cantor believed that there was no essential difference between the potential infinite and the corresponding actual infinite. He reasoned," . . . in truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on."[26]

But wait! We are speaking of infinite sets – sets with an infinite number of elements – without yet defining them. What precisely is an *infinite set*? It was Cantor's genius in answering this question that made possible his revolutionary discoveries.

Suppose a cave-dweller, living long before the invention of numbers, wanted to know whether two rock piles contained the same number of rocks. With no aid from a numbering system, the cave-dweller could resort to the most basic technique possible. The rocks from one pile could be paired one-to-one with the rocks from the second pile. If no rocks remained in either pile after the pairing, then the piles have an equal number of rocks. This is the notion of a *one-to-one (1-1) correspondence*. A 1-1 correspondence is a one-to-one pairing of elements from two sets which exhausts both sets. If there is a 1-1 correspondence between two sets, we say the sets have *equal cardinality*. (The *cardinality* of a set is the number of elements in the set.)

Mathematicians used one-to-one correspondences to compare sizes of sets long before Cantor. However, in applying the technique to infinite sets, some apparent contradictions arose. For example, it was long known that there is a natural 1-1 correspondence ($x \mapsto 2x$) between the intervals [0, 1] and [0, 2] even though the first is a subset of the second. Similarly, there is a natural 1-1 correspondence ($n \mapsto n^2$) between the sets $\{1, 2, 3, . . . \}$ and $\{1, 4, 9, . . . \}$. Galileo noticed this paradox and concluded,

> we can only infer that the totality of all numbers is infinite, and that the number of squares is infinite . . . ; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally, the attributes 'equal,' 'greater,' and 'less,' are not applicable to infinite, but only to finite quantities.[27]

Cantor's stroke of genius came in realizing that the paradox existed only because mathematicians were unwilling to let infinite sets enjoy different properties than finite sets. Since

[25] Tobias Dantzig, p. 211.
[26] Rudy Rucker, p. 3.
[27] Rudy Rucker, p. 6.

this curious property that a set can be put into a 1-1 correspondence with a proper subset of itself occurred only in sets which were considered infinite, Cantor used this property to *define* infinite sets. That is, Cantor defined an infinite set to be a set *which can be put into a one-to-one correspondence with a (proper) subset of itself.*

To get a better feel for this definition, let us consider some examples. Consider the set of natural numbers **N** = {1, 2, 3, . . . } and one of its proper subsets {10001, 10002, 10003, . . . }. Since there is a 1-1 correspondence between these sets, the set **N** of natural numbers is infinite.[28] We can also easily find a 1-1 correspondence between **N** and the set {10, 20, 30, . . . } or the set {1, 4, 9, . . . }, so all these sets have the same cardinality.

Cantor then asked an obvious question: Are there any infinite sets which cannot be put into a 1-1 correspondence with $\mathcal{N}$? A natural suspect was the set $\mathcal{Q}$ of all rational numbers since between any two natural numbers there are infinitely many rational numbers. Cantor's intuition told him that there were more Rational numbers than Natural numbers.[29] But after repeatedly trying unsuccessfully to show that there were more rational numbers than natural numbers, Cantor started having doubts. Eventually he reversed course and tried to prove that the rational numbers have the same cardinality as the natural numbers.[30] At some point he had a flash of brilliance. Here is what he did:

> We make a table which contains all of the positive Rational numbers in the following way: In the first row list all of the Natural numbers $\{1, 2, 3, 4, \ . \ . \ . \ \}$. In the second row, list all of the Rational numbers with a 2 in the denominator: {1/2, 2/2, 3/2, 4/2, . . . }. In the third row, do the same with 3s in the denominators. Continuing in this way, all the positive rational numbers will be included in the table. (Where will 17/25 be?)
>
> Now, let's get rid of the numbers which appear more than once. For example, 2/2 = 1 and 1 is already in the first row, so we erase 2/2 from the second row. Similarly erase 4/2, 6/2, 8/2, ... Which numbers need to be erased from the third row? from the fourth row?
>
> We are almost done. We just have to show that there is a way to pair (i.e.,

[28]Interestingly, John Newton, who wrote *Amazing Grace*, expressed the eternity of heaven in the same way that Cantor defined infinite sets.

[29]Part of the inherent intrigue of Mathematics is that sometimes one's intuition is dead wrong. That leads to neat new surprises and discoveries, which is why proofs are important. A proof verifies what the intuition suggests. Mathematicians often work like the criminal justice system. First the police find the suspect that they think is guilty. But then the prosecutor has to "prove" the guilt by convincing a skeptical jury.

[30]Typical graduate school problems leave it to the student to decide whether the proposition is true or false. While in graduate school I walked clockwise through the hallways when trying to prove a statement true, and counterclockwise while trying to show it to be false.

form a 1-1 correspondence) between the Rational Numbers left in the table and the Natural numbers. The most obvious pairing would be to match 1 with 1, 2 with 2, 3 with 3, and so on. Unfortunately this "pairing" doesn't work. For even though all of the Natural numbers get paired, most of the Rational numbers are left out. For example, nothing get paired with 1/2. Let's try again. (Here comes the brilliant idea.) Instead of going straight down one row of the table, we instead form "rows" down the diagonals. So we will match 1 with 1, 2 with 2, 3 with 1/2, 4 with 3, 5 with 1/3, 6 with 4, 7 with 3/2, 8 with 2/3, 9 with 1/4, and so on.

Thus, we have found a 1-1 correspondence between the positive Rational numbers and the Natural numbers. So, by Cantor's definition, the two sets have the same number of elements.

We have to give Cantor credit for the clever proof, but the downside is that we still don't know if there are infinite sets with different sizes. What would be another good set of numbers to check? How about the Real numbers? That is what Cantor tried.

If Cantor's previous proof was brilliant, his next one was a true stroke of genius! He proved that there are more Real numbers than Natural numbers through a proof by contradiction. That is, he started his proof by assuming that *there was* a 1-1 correspondence between the sets, and then showed that this led to a contradiction. Here is the way the argument goes:

Assume it is possible to find a 1-1 correspondence between $\mathcal{N}$ and the Real numbers between 0 and 1. Now the decimal representation of any number between 0 and 1 will look like $0.x_1x_2x_3x_4\cdots$ where each x_i represents a digit in the decimal representation. So we try the following pairing:

$$\begin{array}{c}
1 \sim 0.x_1^1x_2^1x_3^1x_4^1x_5^1\ .\ .\ . \\
2 \sim 0.x_1^2x_2^2x_3^2x_4^2x_5^2\ .\ .\ . \\
3 \sim 0.x_1^3x_2^3x_3^3x_4^3x_5^3\ .\ .\ . \\
4 \sim 0.x_1^4x_2^4x_3^4x_4^4x_5^4\ .\ .\ . \\
5 \sim 0.x_1^5x_2^5x_3^5x_4^5x_5^5\ .\ .\ . \\
. \quad\quad . \quad\quad . \\
. \quad\quad . \quad\quad . \\
. \quad\quad . \quad\quad .
\end{array}$$

There is our proposed pairing – perfectly general. Does it work? We have used all of the Natural numbers, have we used up all of the Real numbers between 0 and 1? Nope. I can show you a Real number between 0 and 1 which has not been paired with *any* Natural number. It is the number $y = 0.y_1y_2y_3y_4y_5\ldots$ where y_i is chosen to be a digit different from x_i^i. Let's be specific: If $x_i^i < 5$, then we will set $y_i = 8$. If $x_i^i \geq 5$, then we will set $y_i = 2$. Since the decimal representation of y differs from each of the Real numbers listed in at least one decimal place, it must be different from all of them. Thus y is not on the list. By assuming that there was a 1-1 correspondence, we have arrived at a contradiction. Hence the initial assumption (that a 1-1 correspondence existed) was wrong.

Cantor thus concluded that the cardinality of **R** is greater than the cardinality of **N**. Hence Cantor showed there were different sizes of infinity! But how many sizes were there?

A set with three elements, say $\{1, 2, 3\}$ gives rise to $2^3 = 8$ subsets: $\{\ \}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$. Similarly a set with k elements has 2^k subsets. In particular, a finite set has more subsets than elements. Cantor proved this was true for infinite sets as well. That is, no set can be put into a 1-1 correspondence with the set of all of its subsets. Hence the collection of all subsets of $\mathcal{N}$ (which we denote by $\mathcal{P}(\mathcal{N})$) has a different (greater) cardinality than the set $\mathcal{N}$ itself. Let's see why. We again use a proof by contradiction.

Let B be some set (finite or infinite). We now show that $\mathcal{P}(B)$ has a greater cardinality than B.

If $b \in B$, we can pair b with the subset $\{b\}$. Thus, $Card(B) \leq Card(\mathcal{P}(B))$. Now let's assume that *there is* a 1-1 correspondence between B and $\mathcal{P}(B)$. We will call this correspondence f. (So if $3 \sim \{1,2\}$, then we write $f(3) = \{1,2\}$.)

Now, let $A = \{b : b \notin f(b)\}$. (So, e.g., if $f(3) = \{1,2\}$, then $3 \in A$. If $f(3) = \{1,3\}$, then $3 \notin A$.) Then A is a subset of B, so $A \in \mathcal{P}(B)$. Since $A \in \mathcal{P}(B)$, and since f is a 1-1 correspondence, there is some element of B, let's call it b_0, for which $f(b_0) = A$.

Now, Aristotelian logic tells us that either $b_0 \in A$ or $b_0 \notin A$. Let's check to see if either is possible.

If $b_0 \in A$, then $b_0 \notin f(b_0)$ (by defn of A), so $b_0 \notin A$ (since $f(b_0) = A$) which is a contradiction.

If $b_0 \notin A$, then $b_0 \notin f(b_0)$ (since $f(b_0) = A$). Since $b_0 \notin f(b_0)$, then $b_0 \in A$ (by defn of A). This too is a contradiction. Hence neither possibility holds, which is an absurdity.

Since the assumption that there is a 1-1 correspondence between B and $\mathcal{P}(B)$ leads to an absurd conclusion (a contradiction), the assumption must be false. Thus there is no 1-1 correspondence between B and $\mathcal{P}(B)$. That is $Card(B) \neq Card(\mathcal{P}(B))$. Since $Card(B) \leq Card(\mathcal{P}(B))$, it must be the case that $Card(B) < Card(\mathcal{P}(B))$.

Before continuing on with the infinite, let's ponder this proof for a moment. The essence of the proof is: If something is, then it isn't; if it isn't, then it is. As we will see in the next section, Bertrand Russell used essentially the same idea to reveal a crack in the very foundation of mathematics. Russell was kind enough to form a popular version of his argument - it is called the Barber Paradox. A barber (a man) puts a sign in his shop which reads, "I shave all the men in town and only the men in town who don't shave themselves." Why can this claim not be true? Notice how this riddle is similar to the above proof.

Cantor used the Hebrew symbol $\aleph_0$ (aleph-nought) to denote the cardinality of $\mathcal{N}$. Then in analogy with the finite case, he let $2^{\aleph_0}$ express the cardinality of $\mathcal{P}(\mathcal{N})$.

So how many sizes of infinity are there? To get the answer, we need only realize that the process whereby we obtained the set $\mathcal{P}(\mathcal{N})$ from $\mathcal{N}$ can be repeated indefinitely. For example, $\mathcal{P}(\mathcal{P}(\mathcal{N}))$, the collection of all subsets of the collection of all subsets of $\mathcal{N}$, has greater cardinality than $\mathcal{P}(\mathcal{N})$. (In fact, we denote it by $2^{2^{\aleph_0}}$.) Thus THERE ARE AN INFINITE NUMBER OF SIZES OF INFINITY.

Cantor also showed that the cardinality of the Real numbers was equal to $2^{\aleph_0}$, but that led him to ask: Is there a size of infinity between $\aleph_0$ and $2^{\aleph_0}$? Even with all of his genius, this question stumped Cantor. He thought the answer was "no", but was never able to prove it. Mathematicians found the question so perplexing and potentially important that it was posed to the entire mathematical community by David Hilbert in the year 1900 as the first of 23 great unsolved problems to be solved during the 20th century. It took until 1963 to get the full answer – and the answer shocked everyone. How could that be, what is so shocking about either a "yes, there is" or a "no, there isn't" answer. As it turned out, neither of those *were* the answer. The answer turned out to be: It doesn't matter. Either way will work. Either answer will give a consistent and usable mathematics (though different from the other). So in answering the question, one is just coming to a "fork in the road of mathematics" and is forced to go one way or the other. Both roads lead to new mathematical places, but different places from the other path. Given that, we will stop here at the fork, ignoring the advice of Yogi Beara, "When you come to a fork in the road, take it!"

A second intriguing question which bugged Cantor was: Is there a largest infinity? From the previous proof, it would seem there is no "largest infinity." Why? Because given *any* infinite set, one can take the set of all subsets of that set - which we proved is larger in size. Cantor realized this too. On the other hand, Cantor mused – what would be the cardinality of the Set of all Sets? Certainly this "grand-daddy of all sets" cannot be enlarged. Thus its cardinality would have to be the largest of all infinite numbers. How is this dilemma solved? In the same way that ostriches (supposedly) solve their problems: By burying our heads in the sand. Mathematicians today refuse even to consider the Set of all Sets, thus avoiding the paradox of the largest infinity. Cantor himself "solved" the problem by claiming that there exists an infinity which lies above all the rest, but which cannot be approached through an iterative process any more than $\aleph_0$ can be reached by counting through the Natural numbers. Cantor called this supreme infinity the *Absolute infinity* – or just the *Absolute*.

But what does it mean for the Absolute to exist? Well, what does it mean for $\aleph_0$ to exist, or for that matter, what does it mean for the number "5" to exist? Would "5" exist were it not for the presence of a natural world which contained sets with five elements? Somehow the existence of such sets makes us a bit more secure in speaking of "5." (Or maybe the reader doesn't share this paranoia.) Can the same be said for infinity – do sets with an infinite number of elements arise in the natural world? We will reflect on this last question ourselves before considering Cantor's answers to the above questions.

IV. Actual Actual Infinities.

Do we live in an infinite universe? Consider the following questions:

- Is the time dimension infinite?
- Is time infinitely divisible?
- Is space infinite?
- Is space infinitely divisible?
- Is our three-dimensional space one of infinitely many three-dimensional spheres in four-dimensional space?
- Is our three-dimensional space a sphere in a four-dimensional space which is a sphere in a five-dimensional space which is a sphere in a . . .

If the answer to any of the above questions is "yes", then we can think of ourselves as living in an infinite universe. However, none of the above questions has yet been answered. That is not to say that there is no evidence (and strong opinions) on all sides. The point I wish to make is that, as R. Rucker points out, " . . . it is entirely possible that our universe is in every sense finite."[31] Let us briefly consider how this would be possible.

It is now generally agreed that the universe has a finite past – some 13.7 billion years. Depending on the density of the universe, it may have a finite future as well. If the universe is

[31]Rudy Rucker, p. 36.

dense enough, it will collapse back in on itself; if not (as present evidence seems to indicate) it will go on expanding forever.

Secondly, consider the expanse of space. As noted in the introduction, a globe is a finite – but unbounded – surface in three-dimensional space. If one follows a "straight line" (i.e. a great circle) on a globe far enough, one returns to the starting point. Analogously, if Einstein's Theory of General Relativity is correct, it is possible that our universe is likewise a three-dimensional globe in four-dimensional space. Then if one travels far enough along a "straight line" (i.e. a beam of light), again one eventually returns to one's starting place. So even without a boundary, three-dimensional space may still be finite.

But are not time and space at least infinitely divisible? These questions may never be settled, but we need not assume that space and time are infinitely divisible just because they have a smooth feel to them. (Analogously, the motion of a motion picture and the sound from a compact disc appear to be smooth and continuous even though in actuality both are discrete – i.e., filled with gaps.) Support for this point of view comes from Hilbert who speaks to several of the above questions:

> When we turn to the question, what is the essence of the infinite, we must first give ourselves an account as to the meaning the infinite has for reality: let us then see what physics teaches us about it.
>
> The first naive impression of nature and matter is that of continuity. Be it a piece of metal or a fluid volume, we cannot escape the conviction that it is divisible into infinity, and that any of its parts, however small, will have the properties of the whole. But wherever the method of investigation into the physics of matter has been carried sufficiently far, we have invariably struck a limit of divisibility, and this was not due to a lack of experimental refinement but resided in the very nature of the phenomenon. One can indeed regard this emancipation from the infinite as a tendency of modern science and substitute for the old adage *natura non facit saltus* its opposite: Nature does make jumps. . . .[32]

We now reveal Cantor's answers to the questions above and also study the presuppositions that gave rise to his answers.

V. Cantor's Underlying Philosophy

Cantor's answers to the questions concerning the existence of infinite things is contained in the following quote:

> The actual infinite arises in three contexts: first when it is realized in the most complete form, in a fully independent other-worldly being, *in deo*, where I call it the Absolute Infinite or simply Absolute; second when it occurs in the contingent, created world; third when the mind grasps it *in abstracto* as a mathematical magnitude, number, or order type. I wish to make a sharp contrast between the

[32]Tobias Dantzig, p. 237-238.

> Absolute and what I call the Transfinite, that is, the actual infinities of the last two sorts, which are clearly limited, subject to further increase, and thus related to the finite.[33]

The information and theories of the previous section which point to (at least the possibility of) a finite world have come to light since Cantor's time. However, it is doubtful whether they would have influenced Cantor's ideas concerning the existence of infinities. That is because his reasons for believing in natural occurring infinities were tied up with his belief in the existence of numbers.

For Cantor, the existence of any mathematical notions, in particular numbers – finite or infinite – was contingent only upon their being internally logically consistent and being able to fit in with the rest of mathematics. As Cantor said,

> Mathematics is completely free in its development and only bound by the self-evident consideration that its concepts must be both consistent in themselves and stand in an orderly relation fixed through definitions to the previously formed concepts already present and tested.[34]

Referring to numbers in particular, he said,

> In particular one is only obliged with the introduction of new numbers to give definitions of them through which they achieve such a definiteness and possibly such a relation to the older numbers that in given cases they can be distinguished from one another. As soon as a number fulfills all these conditions, it can and must be considered in mathematics as existent and real.[35]

Cantor also acknowledged that the question of existence is tied to the real world:

> Secondly, reality can be ascribed to numbers insofar as they must be taken as an expression or image of the events and relationships of that outer world which is exterior to the intellect. So for instance, the various number-classes (I), (II), (III) etc. are representatives of powers which are actually found in corporeal and intellectual nature. This second species of reality I call the transsubjective or transient reality of the integers.[36]

However, for Cantor the first kind of existence implies the second:

> There is no doubt in my mind that these two forms of reality are always connected with one another. For, a concept said to exist in the first sense also always possesses in certain, even infinitely many, respects a transient reality . . .[37]

[33]Rudy Rucker, p. 10.
[34]Michael Hallett, p. 16.
[35]Ibid., p. 17.
[36]Ibid., p. 18.
[37]Ibid., loc. cit.

Hallett emphasizes,

> As Cantor himself says, what he proposes is a platonic principle: the ‘creation’ of a consistent coherent concept in the human mind is actually the uncovering or discovering of a permanently and independently existing real abstract idea. . . . for Cantor, concentration on coherence was not a means of avoiding discussion of existence but rather a means of guaranteeing it. ‘Coherence’ may be looked upon as a kind of minimal condition that mathematics has to respect, but according to Cantor’s doctrines this minimal condition is an existential maximal principle: as many things as possible exist. Thus in the case of infinite numbers, which occupied him above all, coherent integration of the concept ‘transfinite ordinal number’ guarantees that there must be objects which fall under it.[38]

In other words, Cantor believed that the consistency of infinite numbers implied existence of infinite numbers as a mathematical concept which in turn implied the existence of actual infinities in the natural world. What led him to these bold conclusions? Interestingly, it was his religious beliefs.

Georg Cantor had a religious upbringing which deeply influenced his entire life both personally and professionally. His father, born in Copenhagen, grew up in an evangelical Lutheran mission in St. Petersburg, while his mother was baptized Roman Catholic. His parents were married in the Evangelical Lutheran Church of St. Petersburg, and Cantor, along with his five younger siblings, was baptized as a Lutheran. The spiritual guidance from his father is clearly evidenced by the letters that Georg received during his schooling at Darmstadt and the Polytechnic Institute in Zurich. On the occasion of Cantor’s confirmation his father wrote him a letter encouraging him to keep a truly religious spirit – an unshakable, enduring faith in God – as his truest friend. This was to be his guard against buckling under the resistance that he might encounter in life. Cantor kept this letter with him both physically, and – from all indications – practically his entire life.[39]

Cantor exhibited the extent to which he took his father’s advice in a letter to a fellow mathematician where he wrote,

> My theory stands as firm as a rock;every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things.[40]

[38]Ibid., p. 18-19.

[39]Joseph Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Princeton: Princeton University Press, 1990, p. 272-275.

[40]Ibid., p. 298.

As Hallett explains, "this understanding of what numbers are, or what sets etc. exist, is for Cantor intimately connected with the attempt to understand God's whole abstract creation and the nature of God himself."[41] Cantor indicated that he was deeply influenced by Augustine and Aquinas. For example, Cantor included the extended quote of Augustine (p. 3) in one of his works, and himself wrote (in close similarity to that passage),

> Each individual *finite* cardinal number is in God's intellect both a representative idea and a unified form for the knowledge of innumerably many compound things, that is, those which possess the cardinal number in question. All *finite* cardinal numbers are thus distinct and simultaneously present in God's intellect.[42]

As the above quote indicates, for Cantor the existence of numbers was based on God's ability to perceive them. But how do we know whether God can perceive them? Cantor answers,

> If I have recognized the inner consistency of a concept which points to a being, then the idea of God's omnipotence impels me to think of the being expressed by the concept as in some way actually realizable. Consequently I call the being concerned a 'possible' being. By this is not meant that the being somewhere, somehow, and sometime exists in actuality, since that depends on further factors, but only that it can exist.[43]

That is, Cantor believed that it was *possible* for actual infinities to occur in nature, since he had shown their inner consistency. Moreover, Cantor says elsewhere that "All these particular modes of the transfinite have existed from eternity as ideas in the Divine intellect."[44] However do actual infinities actually exist in nature? Cantor's answer:

> Since God is of the highest perfection one can conclude that it is possible for Him to create a *transfinitum ordinatum.* Therefore, in virtue of His pure goodness and majesty we can conclude that there actually is a created *transfinitum.* . . . the transfinite not only expresses the extensive domain of the possible in God's knowledge, but also presents a rich and continually increasing field of ideal discovery. Moreover, I am convinced that it also achieves reality and existence in the world of the created, so as to express more strongly than could have been the case with a mere 'finite world' the majesty of the Creator following his own free decree.[45]

We noted above that Cantor was heavily influenced by Aquinas. In fact, Cantor agreed with the logic of Aquinas' argument for a finite world (p. 4). Yet Cantor came to the opposite conclusion. How could this be? Aquinas had argued that actual infinities do not exist in

[41] Michael Hallet, p. 10.
[42] Ibid., p. 36.
[43] Ibid., p. 20.
[44] Ibid., p. 21.
[45] Ibid., p. 23-24.

the natural world because God numbered His creation and only finite numbers existed. But Cantor points out that he has now shown that infinite numbers also exist (based on their logical consistency), and so they too are certainly available to God. It appears then that Cantor would have expected Aquinas to have agreed with him, had Aquinas seen Cantor's development of the transfinite numbers.

In summary, Cantor believed that since he had shown the concept of infinite numbers to be logically consistent both internally and in relation to the rest of mathematics (in particular the finite numbers), it followed that God fully understood infinite numbers. Two inferences followed from God's knowledge of the infinite; first, the infinite numbers *do exist*, and secondly, God has used some of the infinite numbers in creation to show His majesty. That is, actual infinities do exist in nature.

Finally, we come to the Absolute Infinite, or simply the Absolute. Recall that the Absolute was conceived by Cantor in an effort to resolve the paradox caused by 1) Cantor's proof that any infinite set gives rise to a larger infinite set, and 2) his belief that the set of all sets necessarily must represent the largest possible infinity. Cantor's description of it includes the following:

1. The Absolute "transcends the human power of comprehension, and in particular is beyond mathematical determination."[46]

2. "The Absolute can only be acknowledged and admitted, never known, not even approximately."[47]

3. The Absolute "cannot in any way be added to or diminished."[48] As such, the Absolute is qualitatively different from all (other) infinities.

4. The Absolute cannot be approached from below through a rational constructive process. It is "unreachable by any determination."[49]

It is readily apparent that the above description of the Absolute has a theological "ring" to it. Writers of scripture describe God in a similar fashion. For example, Isaiah records (Isaiah 55:8-9), "For my thoughts are not your thoughts, neither are my ways your ways, says the Lord. For as the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts." And Paul writes to the church in Rome (Romans 11:33-34), "O the depth of the riches and wisdom and knowledge of God! How unsearchable are his judgments and how inscrutable his ways! 'For who has known the mind of the Lord, or who has been his counselor?'" We also find, " . . . for I am God, and there is

[46]Ibid., p. 13.
[47]Ibid., loc. cit.
[48]Ibid., loc. cit.
[49]Ibid., p. 39.

no other; I am God, and there is none like me . . . " recorded in Isaiah 46:9. Other scriptures that attest to the transcendence and singular uniqueness of God include Job 11:7-8; 22:22; 36:26; 37:5,23; Psalms 145:3; Ecclesiastes 11:5; Isaiah 40:28; Malachi 3:6; Matthew 11:27; and I Corinthians 2:11,16. Furthermore, 4) is closely paralleled by St. Gregory's thought,

> No matter how far our mind may have progressed in the contemplation of God, it does not attain to what He is, but to what is beneath Him.[50]

The similarities between the descriptions of God and the Absolute are not coincidental. Cantor thought of the Absolute as representing the very knowledge of God. Hallett explains that,

> The most natural way to interpret the Absolute . . . is not as something mathematizable itself but as the category of everything mathematizable . . . [so] to mathematize the Absolute would be simply a category mistake: everything mathematizable (or numerable) is already *in* the realm of the finite and transfinite, and the absolute is simply that which embraces all these. . . . For if the Absolute is taken to represent the category of all mathematical forms, it represents then . . . 'the extreme domain of the possible in God's knowledge'. . . . as Kowalewski described it: 'these powers, the Cantorian alephs, were for Cantor something holy, in a certain sense the steps which led up to the throne of the infinite, to the throne of God.'
>
> This association of the Absolute with God makes it finally clear why the Absolute cannot be subjected to any attempt at rational (or in particular, mathematical) understanding. This is a permanent and ineradicable imperfection or gap in our understanding. We may have a feeling or an inkling about Absoluteness or God, but these feelings can never be explicated in any clear intellectual sense. This is what Cantor means when he talks of 'the true Absolute, which is God, and which permits no determination'.[51]

With the Absolute Infinite, Cantor has taken us, as it were, full circle. The actual infinite had been an enigma for theologians, philosophers and mathematicians alike for centuries. Cantor not only clearly defined it, but also showed it to be both self-consistent and consistent with the rest of mathematics. Moreover, he proved that there were (infinitely many) different sizes of infinity, and even showed how these infinite numbers could be manipulated algebraically (transfinite set theory). In so doing, Cantor removed the mystic aura surrounding the infinite and brought it totally within the embrace of mathematics. However, Cantor's own set theory eventually led him to the Absolute Infinite which, being "beyond mathematical determination," again crossed over the boundaries of mathematics into philosophy and theology.

[50]Rudy Rucker, p. 46.
[51]Michael Hallett, p. 43-44.

Maybe it is fairer to say that Cantor's accomplishments spiraled the discussion of the infinite up to a higher plane. Although the mathematical community was slow to accept his ideas, Cantor's theory is now not only firmly established as its own field of interest, but also is indispensable for much of the rest of mathematics. According to Joseph Dauben who wrote the definitive biography of Cantor,

> [Cantor's] transfinite set theory represented a revolution in the history of mathematics. Not a revolution in the sense of returning to earlier starting points, but more a revolution in the sense of overthrowing older, established prejudices against the infinite in any actual, completed form. Consequently, Cantor's transfinite numbers were to prove no less revolutionary for philosophers and theologians who were concerned with the problem of infinity. . . . Cantor was deeply committed to probing the metaphysical and religious significance of his work. In fact, for him the mathematical, metaphysical, and theological aspects of his transfinite set theory were mutually reinforcing. Cantor was convinced that his discoveries were not only essential for the future of pure mathematics but that set theory could be used to refine philosophy and to support theology.[52]

VI. Implications of Infinity

We finish by considering how the infinite has, indeed, left its imprint on the physical, rational, and spiritual dimensions of our lives. We are physical beings that live and move in a physical world. It is natural and right that our knowledge of the world inform our actions as we live in it. As we saw earlier, it is entirely possible that we live in a finite world. How might this influence our behavior? It seems that our past and present careless use and abuse of natural resources comes at least in part from false, wishful thinking that the land, oceans, and atmosphere have infinite extent. Certainly (we like to think) any great body of water can absorb whatever we dump into it, our supply of trees and ground water and fossil fuels is unlimited, air pollution is just a local problem that gets blown away. Only in the honest realization that this vast life-support system we call earth is indeed limited, will we begin behaving properly towards it. As we begin to come to grips with the realization that we are living on a finite planet, we have begun to explore the heavens. What is to be our attitude toward the – possibly finite – cosmos? Only time will tell.

Time poses similar challenges. We do not yet know whether or not the universe has an infinite future, but we can be fairly certain that each of us as individuals has but a finite time to spend on this earth. How does this affect us? It leads some to seek the fountain of youth in various forms. Harold Kushner points out in his book *When Bad Things Happen to Good People* that it is the very finiteness of one's lifetime that makes time valuable. C. Keyser, an early twentieth century mathematician agrees:

[52] Joseph Dauben, p. 118.

> This finiteness of life, its temporal finitude, is, for our quest, a fact of supreme importance. For, owing to a radical insight of modern science, we know at length that the distinction of finite and infinite is very profound, deep as the nature of being, cleaving the world . . .
>
> The thesis, which is simple, is not remote or difficult to grasp. It is that the values of life are values characteristic of mortal life; it is that the temporal finitude of life is essential to its worth; it is that, were it not for death, if life did not end, if it were a process of infinite duration, it would be devoid of the precious things that make us yearn for its everlasting perpetuation. . . .
>
> Our human speech, if it were without the tender and sacred things it has from death, would ill befit a life-loving race of mortals. The precious associations that cluster about the words, friendship, love, husband, wife, father, mother, parent, child, brother and sister, youth and age, flourish and bloom only in the heart of a life that begins and ends. But it is not only in these common goods of life, not only in the humbler joys that bless us all from day to day, that the spiritual significance of mortality may be discerned. More subtly indeed but not less certainly is it manifest also in the austere felicities of the higher reason: visions of the infinite, the swift march of time, the irrevocableness of the past, the eternality of truth, the inexorableness of cosmic law, the imperturbability of nature's gaze upon the struggles and strifes of men, the unbroken silence of destiny – all these solemn beatitudes of reason and meditation derive their poignance from the transitoriness of the life that contemplates them. Nay, whatsoever is dearest in the sphere of *outer* sense – the beautiful garment of the external world, the wondrous drama of the revolving year, alternation of day and night, of morning and even-tide, ocean's voice, and 'the rhythmic sighing of the wind'; whatsoever is highest and holiest in the sphere of *inner* sense – the tenderness and piety of art, the mellowness of wisdom, the serenity and peace of renunciation, charity, mercy, and service: *all* of the sacred values that constitute life a priceless boon are subtly bred in the all-pervasive sense of its temporal finitude. Death is not the tragedy of life; it is a limitation of life, essential to its beatitudes: the tragedy is that, if it were not for death, life would be void of worth.[53]

The infinite has also touched our lives on the rational level by changing the way we think about truth. We will show how the infinite has been the "clay feet" of man's statue of truth. Indeed, Joseph Dauben states, "The question of infinity had brought mathematicians to the edge of uncertainty."[54]

At the trial of Jesus, Pilate asked rhetorically, "What is truth?" If a philosopher of the times had responded in the same way that former Supreme Court Justice Potter Stewart answered the question "What is obscenity?" (i.e., "I can't define obscenity, but I know it when I see it."), then he may have replied: "Consider geometry – there is an example of truth." Remarkably, geometry as developed by Euclid in the *Elements* some three centuries

[53] Cassius J. Keyser, *The Rational and the Superrational*, New York: Scripta Mathematica, 1952, p. 123-125.

[54] Joseph Dauben, p. 266.

earlier, would be considered the paradigm of truth for the next eighteen centuries as well. For certainly, if truth was to be found, it would be found in mathematics, and if in mathematics, most assuredly in geometry. After all, geometry was based on five postulates – self evident truths – of the physical world, and extended to 465 theorems (proved statements) using the infallible deductive method of proof. Moreover, the truthfulness of each of the theorems was substantiated by empirical evidence – the world of the surveyor coincided exactly with the world of the logician/geometer. It was, in fact, this complementary witness of truth for which the human mind seems to yearn – a witness from both the inside and the outside – which made the truth of geometry seem irrefutable.

What were these five postulates on which the truths of Euclid's geometry rested? They are:

1. Given two distinct points, exactly one straight line segment can be drawn between them.
2. Any line segment can be extended indefinitely in only one way.
3. A circle can be drawn with any center and with any radius.
4. All right angles are equal.
5. Given any line and a point not on that line, there is exactly one line passing through that point which will never intersect the first line - even though indefinitely extended.

Don't these seem obvious? Euclid did not prove any of these - he laid these down as the foundational starting point. These were truths which could not be proved, they just had to be accepted as true. Does knowledge of these five postulates depend on experience, or are they known *a priori*?[55]

As noted above, once these postulates are accepted, they could be used to mathematically prove many other geometric results. Now in order to build a mathematical argument, other working assumptions were needed as well. Assumptions such as: 1) Things that are equal to the same thing are equal to each other, and 2) If two things are equal, and equal things are added to them, then what results are also equal. But these certainly seem safe.

So, starting with a seemingly firm foundation and using an equally secure process, Euclid proved 465 geometric theorems. Then, when he gave these theorems to surveyors and builders to use, everyone came back satisfied. There were no discrepancies - hence no second thoughts. Is it any wonder that Euclid's Geometry was considered for over two millennia the prime, unquestionable example of pure truth.

[55] *a priori* means roughly "independent of experience," or "without relying on information coming to us through the five senses."

Plato argued that the truths of geometry were universal, immutable, and not learned from experience. Rationalistic philosophers such as Descartes and Spinoza agreed that the mind can perceive truths *a priori*, and geometry was held up as indisputable evidence. (E.g., Spinoza's favorite example of an undeniably true statement was that the interior angles of a triangle sum to a straight line.)

Rationalism (which holds that we know fundamental truth *a priori*) was challenged by John Locke and other empiricists (who believed knowledge comes via our senses), but even they differentiated mathematics from the rest of knowledge which could only be learned from experience. Finally, there was Immanuel Kant who wrote in *Prolegomena to Any Future Metaphysics*,

> We can say with confidence that certain pure a priori synthetical cognitions, pure mathematics and pure physics, are actual and given; for both contain propositions which are thoroughly recognized as absolutely certain . . . and yet as independent of experience.[56]

Likewise, he affirmed in his *Critique of Pure Reason* that all of the axioms and theorems of mathematics were truths.

However, just as Kant was writing those words, there was appearing on the horizon "a small cloud no larger than a man's hand." Later, in hindsight, Morris Kline, a mathematician and historian, could say of Kant:

> His inability of conceive of another geometry convinced him that there could be no other. . . . Kant's boldness in philosophy was surpassed by his rashness in geometry, for despite never having been more than forty miles from his home city of Konigsberg in East Prussia, he presumed he could decide the geometry of the world.[57]

What was this "small cloud?" For centuries, generations of mathematicians had tried to simplify Euclid's geometry by getting rid of the Fifth Postulate. This postulate, commonly known as the Parallel Postulate, has a subtle reference to infinity because it refers to lines being "indefinitely extended." This made it less self-evident than the other four, hence less acceptable. One way to remove it as an assumption was to derive it from the other four (thus, making it a theorem). Girolamo Saccheri tried this approach in the early eighteenth century by assuming the Fifth was wrong and hoping that the deductive process would lead to an absurdity. (Does this kind of argument ring a bell - it is again the proof by contradiction method.) Indeed, Saccheri did show that by denying the Fifth postulate one arrived at strange conclusions – e.g. the sum of the interior angles of a triangle depends on the size of

[56] Morris Kline, *Mathematics*, New York: Oxford University Press, 1980, p. 75.
[57] Ibid., p. 76.

the triangle. This certainly seemed to be an absurd conclusion, and was all the convincing he needed to conclude that the Parallel Postulate was, in fact, true.

But it wasn't enough for Karl Friedrich Gauss (1777-1855). The Prince of Mathematics realized that even though Saccheri's strange conclusions did not coincide with our perception of the real world, they may yet be mathematically consistent. *Thus Gauss separated the logical consistency of a mathematical system from its agreement with the physical universe.* With this bold new realization the dam broke, and many others, including Gauss' student, Bernard Riemann (1826-1866), developed other logically consistent geometries by starting with a different set of postulates.

For example, what if we change the fifth postulate to say that, "Given a line and a point not on the line, there *is no* line passing through the point and parallel to the line." Does this "non-Euclidean geometry" make sense?

It not only makes sense, but we are living on it. Instead of thinking of a drawing surface as being a big flat plane (as Euclid did), consider the world of an ant crawling on a beach ball (which is of course the human situation living on earth as well). For Euclid, a straight line was the result of not turning one way or the other, but marching straight ahead. What happens when an ant does that on a ball, or a person on the earth? Equivalently, a straight line on the plane is the shortest distance between two points. What if a piece of yarn or string is stretched taut from one point on a sphere to another? One would get a "straight line" on a sphere. This explains why a flight to Europe, for example, takes a route so far north. Tighten a piece of string on the globe – with one end at Grand Rapids and the other at Rome – and see what happens.

Using this notion of straight lines, straight lines on a sphere are "great circles." That is, they are the largest-possible circles (like the circumference) on a sphere. An alternative definition: Centers of great circles coincide with the center of the sphere. Check that in this geometry (unlike Euclid's) *any* two lines *will* intersect.

How else does the Spherical geometry differ from Euclid's? As was alluded to above, one of Euclid's theorems was that the interior angles of a triangle sum to 180^o. Consider the triangle on the globe consisting of two perpendicular longitudinal lines from the north pole to the equator, and then the portion of the equator between them as the third side. All three of the interior angles are 90^o, making the total 270^o. So much for Spinoza's undeniably true statement that the interior angles of a triangle sum to a straight line! Because of this property of a sphere, the geometry of a sphere is said to have "positive curvature" whereas the plane on which Euclid based his geometry has "zero curvature."

What's the next obvious question? There are surfaces with zero curvature, and with positive curvature, is there a surface with negative curvature - a surface where the the angles of a triangle sum to *less than* 180^o? Indeed there is. It looks like a saddle. Interestingly, this surface has the property that given a line and a point not on the line, there are *infinitely many* lines passing through the point and also parallel to the given line!

The realization that there were geometries different from Euclid's revolutionized our understanding of the relationship between consistency, experience, and truth. Certainly, if a body of knowledge is true, it will both be logically consistent (i.e., contain no internal contradictions) and it will fit with experience. But what about the converse? A board game can be logically consistent, but it is not considered to be truth. On the other hand, a scientific or mathematical idea can never be proved true by experience or experimentation – it can only be proved false by a negative result. So neither internal consistency nor external evidence by itself guarantees truth. However, what happens when a logically consistent mathematical theory (such as Euclid's) is verified by experience as well. Though the criteria were not consciously applied, for two millennia mathematicians considered Euclid's geometry to be (the one and only) truth because of this combination of internal and external support.

The development of new geometries showed that truth cannot be so easily ascertained. In fact it took the question of truth out of the picture. That is not to say there is no truth, but that truth – if it does exist – cannot be determined merely by checking empirical evidence or logical consistency. Consequently, modern scientists tend not to think of their theories as being true or false. Instead they are thought of as models which can be used to predict how nature will behave. If a new model gives more accurate predictions, then it replaces the old one.

Does there exist a model which, if found, would predict all future events perfectly? That is, is there a model which is TRUE? Scientists who believe that a true model exists, think of our present models as approximating the true one. Scientists who don't believe in a true model see nature itself as the asymptotic limit of their efforts. But whether there is a truth toward which research progresses is itself a question which lies outside science and mathematics.

The truth of Euclid's geometry had been assumed since it rested on a twin foundation of logical consistency and empirical evidence. Gauss kicked out half of the foundation when he realized that other geometries could be consistent as well. It was left for Albert Einstein to finish the job. On May 29, 1919 a solar eclipse provided the right conditions to test Einstein's Theory of General Relativity. Einstein's model – based on Riemann's geometry in which space has a variable curvature – showed itself to be a better predictor of natural events. A new generation of scientists has now grown up accepting the fact that we may not live in a Euclidean universe. In fact, NASA is presently in the process of determining the shape of the universe by finding whether the universe has positive, negative or zero curvature.

Thus Euclid's geometry, the paradigm of truth, which stood for centuries like a lighthouse on the shore, was toppled into the sea – the Parallel Postulate, with its suggestion of the infinite, leading to its downfall. The resulting waves have rippled through other disciplines, changing the world.

Kline sums up the situation well:

> All people, prior to non-Euclidean geometry, had shared the fundamental belief that man can obtain certainties. The solid basis for this belief had been that man had already obtained some truths – witness, mathematics. No system of thought has ever been so widely and completely accepted as Euclidean geometry . . . Men such as Plato and Descartes were convinced that mathematical truths were innate in human beings. Kant based his entire philosophy on the existence of mathematical truths. But now philosophy is haunted by the specter that the search for truths may be a search for phantoms.
>
> The implication of non-Euclidean geometry, namely, that man may not be able to acquire truths, affects all thought. Past ages may have sought absolute standards in law, ethics, government, economics, and other fields. They believed that by reasoning one could determine the perfect state, the perfect economic system, the ideals of human behavior, and the like. The standards sought were not just the most effective ones, but the unique, the correct ones.
>
> Our own century is the first to feel the impact of non-Euclidean geometry because the theory of relativity brought it into prominence. It is very likely that the abandonment of absolutes has seeped into the minds of all intellectuals. We no longer search for the ideal political system of ideal code of ethics but rather for the most workable.[58]

Notice that the story of the discovery of non-Euclidean geometries has a close parallel – also involving infinity. Recall Cantor's Continuum Hypothesis which asked whether there was a size of infinity between $\aleph_0$ and $2^{\aleph_0}$. In the same way that Euclid's Fifth Postulate was shown to be neither true nor false – substituting a different postulate just produced a different geometry – Kurt Gödel (1906-1978) showed in 1938 that the answer to Cantor's question can be either yes or no. The answer is independent of the rest of set theory – so one is free to choose whichever answer one likes. The different choices will lead to different (yet equally consistent) mathematics.

So in both of these areas - geometry and set theory which lies at the very foundation of modern mathematics, the infinite has shaken up our naive understanding of truth in mathematics. Dauben concludes,

[58]Morris Kline, *Mathematics for the Liberal Arts*, Addison-Wesley, 1967, p. 475-476.

> Cantor's set theory had brought mathematicians to a frightening and perilous precipice. Cantor's infinite had shaken the traditional faith in mathematics' everlasting certitude . . .[59]

Indeed, the intrigue goes even deeper. Although geometry had been a tightly wrapped subject - all theorems carefully deduced from the five postulates - for over 2000 years, the rest of mathematics had been a hodgepodge. Thus in the late nineteenth century mathematicians tried to find a solid foundation on which all of mathematics could rest. The idea of sets seemed to be the most suitable starting point. In 1893 the German mathematician, Gottlob Frege, began building the rest of mathematics from set theory. Just as he was finishing in 1902, Bertrand Russell sent Frege a letter in which he asked Frege a question concerning sets which amounted to the "If it is, then it isn't; if it isn't, then it is" riddle. Frege realized that Russell's simple question kicked the foundation out of his entire 10 year work. Although Frege published his results, he included the melancholy note, "A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr. Bertrand Russell as the work was nearly through the press."

Russell was by no means heartened by his discovery of a flaw in Frege's logic. Indeed, no one wanted to find a solid foundation for mathematics more than Russell, who set to work with Alfred Whitehead on his own monumental work, *Principia Mathematica*. This unreadable classic which builds mathematics from logic takes over 300 pages to finally prove that $1 + 1 = 2$ and took ten years to write. Although he found a way to mend the flaws in the foundation which he had earlier revealed, it resulted in a mathematics which was less beautiful and intuitive. This led the great twentieth century mathematician, philosopher, moralist, war-protester Russell to lament:

> I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.[60]

[59] Joseph Dauben, p. 270.

[60] from "Portraits from Memory"

"What is truth?" Mathematics for two millennia proudly gave answer, but now is humbly silenced – by the infinite.

Finally, we consider the oft made claim that God is infinite. The infinity of God is often invoked by writers and speakers in order to lend support for a supposed implication of divine infinity. An example of this comes from William Barclay's *Testament of Faith*. Barclay asks the question "But why in the end had Jesus to die?" He answers that it was necessary for Jesus to go to the Cross to show the extent of His love:

> Jesus reveals to men the illimitable, the unconquerable, the literally infinite love of God. . . . The pain and the agony of the Cross was the price that Jesus had to pay, the sacrifice he had to make to set before men the love of God.[61]

Notice that Barcley's argument rests, at least in part, on the assumption that the concept of "the literally infinite love of God" is both meaningful and true.

But are such inferences valid? The question we raise is not whether the statement "God is infinite" is true, but whether it is meaningful. Already we have seen Weyl, Augustine, Aquinas, Locke and Cantor claim that God was infinite – yet each was saying something different in making the statement. To say that God is infinite could be referring to some attribute of God such as His omnipresence (Locke), or it could be making an absolute statement about God's being (Aquinas), or it could be contrasting God's nature with human nature (Weyl), or it could be linking God's knowledge to the set of (natural) numbers (Augustine), or it could be associating the mind of God with the Absolute Infinite (Cantor). Other possibilities (using the dictionary definitions) include meaning that God is unbounded, immeasurably large, or has endless duration.

The statement, "God is omnipresent," for example, is precise and unambiguous since the word "omnipresent" has a single meaning. Thus whatever is implied by being omnipresent, would be true of God. However, since the word "infinite" has multiple meanings, inferences from the statement "God is infinite" are not automatic. This problem arises generally. For example, consider the argument: The set of real numbers between 0 and 1 ([0, 1]) is infinite and infinite means being unbounded, so the set [0, 1] is unbounded. The conclusion is false. The problem is that two different meanings of the word "infinite" were used.

Similarly, saying that God is infinite is ambiguous. This need not keep one from making the claim, but it should make one wary of drawing inferences from the statement as is often done. That is, arguments of the form "God is infinite, therefore . . ." should be viewed skeptically. This need not limit our description of God, since any particular meaning

[61] William Barclay, *Testament of Faith*, Oxford: Mowbray, 1975, p. 51.

of infinity which might be applied to God can still be expressed with other words – e.g., omniscience, omnipotence, omnipresence, and singular uniqueness.

An example where an infinity argument concerning God is used more cautiously comes from Cassius Keyser whose perspective on death we considered earlier. Keyser argues that the academic world is mistakenly holding onto the antiquated notion that *the whole is greater than the part.* This, he explains, is true without exception *in the world of the finite*, however

> the other component – the world of infinite – is composed of wholes for which, without exception, the proposition is false; the discovery that the latter world, the world of infinite wholes, is *par excellence* the domain of reason, and that, in respect of content, it is immeasurably richer than the world of finite wholes: that discovery I judge to be second in importance, for the future of thought, to no event in the history of mankind.[62]

He continues with an example:

> Not long ago in a western city of the United States a great orator, speaking on the dogma that the persons of the Trinity are each Almighty and yet together constitute but *one* Almighty, speaking of the doctrine that each of the Persons is equal to the One composed by all of them, evoked general applause from a vast audience by characterizing the venerated creed as 'infinitely absurd'.

Keyser then shows that when dealing with infinite sets, "three" can constitute "one." Consider the sets A={1, 4, 7, 10, . . . }, B={2, 5, 8, 11, . . . }, and C={3, 6, 9, 12, . . . }. As we have already seen, A, B and C all have the same cardinality as the natural numbers **N**={1, 2, 3, 4, . . . }, yet the union of A, B and C together equals the natural numbers.

Keyser, in a blend of humility and pride, then concludes:

> Have we proved that there is a Trinity composed of three components related to one another and to the Trinity as the dogma asserts? No. We have proved that the *conception* of such a Trinity, instead of being rendered absurd by a so-called axiom having no application to infinite manifolds, is rigorously thinkable, perfectly possible and rational, and that our brilliant orator was indeed in this instance an ass.

The most important point that Keyser makes is that (as we have seen) the infinite is qualitatively different from the finite. It obeys other rules – rules that often do not follow our intuition. This raises the question: Since our daily comings and goings occur in a finite world, and since the infinite, being qualitatively different from the finite, is anything but natural and intuitive, how is it that we have even discovered the concept? Moreover, why are we inclined to dwell on it; why the fascination?

[62] Cassius Keyser, p. 99-101.

Maybe it is because we actually do live in an infinite universe, so our mind's attempt to understand infinity is just part of its attempt to understand our world. We might then agree with Stephen Weinberg, a Noble-laureate in physics, who ends his classic *The First Three Minutes* saying,

> The more the universe seems comprehensible, the more it also seems pointless.
> But if there is no solace in the fruits of our research, there is at least some consolation in the research itself. . . . The effort to understand the universe is one of the very few things that lifts human life a little above the level of farce, and gives it some of the grace of tragedy.[63]

If the physical universe is all there is, then it could be argued that our tendency to believe in a supernatural being is just a perversion (or exploitation) of our sense of the infinite.

On the other hand, maybe, as Thomas Aquinas argued (p. 4), our sense of the infinite implies the existence of an infinite being. In this case our fascination, yet lack of full understanding, could be the result of "seeing through a glass darkly."

Finally, it is possible that our intrigue with the infinite is the result of a purposeful imprint on our minds by the Infinite One. For as we noted earlier, grappling with the infinite, while a testimony to the dignity we possess, produces a certain humility by clarifying our relationship to our world and our Creator. Isn't this what the writer of Ecclesiastes meant when he wrote, "He has put eternity into man's mind . . . "

[63] Steven Weinberg, *The First Three Minutes*, New York: Basic Books, 1977, p. 155.